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Plethysm of S-Functions

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PLETHYSM OF S-FUNCTIONS

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CONTENTS

The first problem solved here is that of determining the coefficient of any S-function $\{\lambda\}$ in the expansion of $\{m\}\otimes\{\mu\}$, where m is a positive integer and (μ) is a partition of 4. The method is not recursive or laborious and can be applied equally well to large as to small values of m. It will also yield a specific formula for the coefficient when $\{\lambda\}$ has any prescribed form. Such formulae given here include those for the coefficients of

$$\{4m-k,\,k\},\quad \{m+k,\,m+k,\,m-k,\,m-k\},\quad \{m+k,\,m,\,m,\,m-k\},\quad \{4m-2k,\,k,\,k\},$$

where $m \geqslant k \geqslant 0$, and several other types.

Some of the results proved incidentally in the development of this method are also of some intrinsic interest. Thus a formula is obtained and proved for the coefficient of $\{\lambda\}=\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ in $\{m\}^n$ when $\lambda_2 \leqslant m$ and n is any integer, and it is proved that the coefficient of any $\{\lambda\}$ in $\{m\}^n$ is congruent to $1, 0$ or -1, mod n when n is prime and is congruent to $1, 0$ or -1, mod $n-1$ when $n-1$ is prime.

From the results obtained for (μ) a partition of four certain S-functions are seen to have the same coefficient in $\{m\}\otimes\{\mu\}$. Thus, if $\lambda_1 \leqslant 2m$, then the coefficients of

$$\{\lambda_1,\,\lambda_2,\,\lambda_3,\,\lambda_4\}\quad\text{and}\quad\{2m-\lambda_4,\,2m-\lambda_3,\,2m-\lambda_2,\,2m-\lambda_1\}$$

are proved to be equal, and if β is even, $\lambda_2 \leqslant m$, and $m+\lambda_4 \geqslant \beta \geqslant \lambda_2$ then the coefficients of

$$\{\lambda_1,\,\lambda_2,\,\lambda_3,\,\lambda_4\}\quad\text{and}\quad\{8m-3\beta-\lambda_1,\,\beta-\lambda_4,\,\beta-\lambda_3,\,\beta-\lambda_2\}$$

are equal.

These results on related coefficients are gathered into five main theorems which are proved for all m and n in the last section of the paper.

1. Introduction

The problem of expressing an invariant matrix of an invariant matrix as a direct sum of irreducible invariant matrices is that of expressing the plethysm $\{\lambda\}\otimes\{\mu\}$ of two S-functions as a sum of S-functions. Several attacks (Littlewood 1944, 1951; Murnaghan 1951 *a*, *b*, *c*;

Robinson 1949, 1950; Thrall 1942; Todd 1949) have been made on this problem, varying considerably in their generality and the degree to which the results obtained have been applicable to numerical cases. As the weights of $\{\lambda\}$ and $\{\mu\}$ increase it becomes more and more apparent that there is need to develop a technique of finding the coefficient of any given S-function in a plethysm, rather than a method, inevitably very laborious, of determining the full expansion. Furthermore, there is need to consider the structure of any given plethysm to see whether the terms can be grouped into sets in any systematic way, or whether any relations exist between the coefficients that arise. Something of this sort is known to be the case for $\{m\}\otimes S_r$ and $\{1^m\}\otimes S_r$ (Foulkes 1951), in which every S-function with k parts appearing in the latter result can be characterized by a partition of m together with a set of k non-negative integers.

Explicit results for $\{m\}\otimes\{\mu\}$, where m is an integer and (μ) is a partition of 2 or 3, are known (Littlewood 1944; Thrall 1942). A method has been given (Duncan 1952 *a*) for (μ) a partition of 4, but it involves S-function multiplication and will entail substantial computation in all but the simpler cases. In the present paper the initial objective is to give a rapid method for the explicit determination of the coefficient of any given $\{\lambda\}$ in $\{m\}\otimes\{\mu\}$, where (λ) and (μ) are respectively partitions of $4m$ and 4. Such a method, exceedingly simple in its application to numerical cases, is obtained. No S-function multiplication is needed, nor recursive relations, and the method is as easy to apply to large values of m as to small values. Some of the incidental theorems which contribute to this method for $n=4$ are proved for the general case when (μ) is a partition of any positive integer n. Among these, one of particular interest is a simple formula for the coefficient of $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ in $\{m\}^n$ whenever $\lambda_2 \leqslant m$.

The precise method developed here for $n=4$ leads to specific formulae for the coefficients arising in $\{m\}\otimes\{\mu\}$. Several of these formulae are given and others can be easily derived. Furthermore, the results for $n=4$ show that various relations, most of them hitherto unsuspected, exist between the coefficients. Thus, if $\{\lambda\}=\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$ and $\lambda_1 \leqslant 2m$, then the coefficients of $\{\lambda\}$ and $\{2m-\lambda_4, 2m-\lambda_3, 2m-\lambda_2, 2m-\lambda_1\}$ in $\{m\}\otimes\{\mu\}$ are equal. Another of these relations is that if $\lambda_2 \leqslant m$ and m is even, then the coefficients of $\{\lambda\}$ and

$$\{m+\lambda_2+\lambda_3+\lambda_4,\ m-\lambda_4, m-\lambda_3, m-\lambda_2\}$$

in $\{m\}\otimes\{\mu\}$ are equal, whereas if m is odd, the coefficient of either of these S-functions in $\{m\}\otimes\{\mu\}$ is equal to the coefficient of the other in $\{m\}\otimes\{\tilde{\mu}\}$, where (μ) and $(\tilde{\mu})$ are conjugate partitions.

In the last part of the paper the various relations observed to hold between the coefficients in $\{m\}\otimes\{\mu\}$, where (μ) is a partition of 4, are proved to hold when (μ) is a partition of any integer n. Apart from their intrinsic interest, these results should be very useful in any further computative work on plethysm of S-functions.

The principal technique employed is the method of differential operators associated with S-functions (Foulkes 1949). My thanks are due to Dr D. G. Duncan of the University of Arizona for sending me his evaluations of $\{7\}\otimes\{4\}$ and $\{7\}\otimes\{21^2\}$ which were useful in verifying some of the theorems, and to Professor A. C. Aitken, F.R.S., for reading the manuscript.

Murnaghan (1951 *c*) has verified the author's determinations of $\{6\}\otimes\{4\}$ and $\{5\}\otimes\{4\}$ (Foulkes 1950), and Littlewood (1944) has given $\{4\}\otimes\{4\}$, $\{3\}\otimes\{4\}$ and $\{2\}\otimes\{4\}$. All the main results of this paper will yield further results by applying the theorem of conjugates (Littlewood 1944), but such derived theorems are easily written down and will not be enumerated here.

2. Preliminary

It is known (Zia-ud-Din 1936; Murnaghan 1951*a*; Foulkes 1949) that if (μ) is any partition of 4, then

$$\{m\}\otimes\{\mu\} = \frac{1}{4!}[\chi^{(\mu)}_{1^4}\{m\}^4 + 6\chi^{(\mu)}_{1^2 2}\{m\}^2\{m\}^{(2)} + 8\chi^{(\mu)}_{13}\{m\}\{m\}^{(3)} + 3\chi^{(\mu)}_{2^2}\{m\}^{(2)}\{m\}^{(2)} + 6\chi^{(\mu)}_{4}\{m\}^{(4)}],$$

where $\chi^{(\mu)}_\rho$ is the characteristic of the class ρ in the irreducible representation of the symmetric group corresponding to (μ), and $\{m\}^{(r)}$ is the expression obtained from $\{m\}$ when every S_k is replaced by S_{kr}, this expression being equal to $\{m\}\otimes S_r$. The S-functions appearing in $\{m\}\otimes\{\mu\}$ are known to be those associated with partitions of $4m$ into not more than four parts. If $(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is a partition of $4m$, where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$, and D_λ is the operator associated with (λ), then it is known that $D_\lambda[\{m\}\otimes\{\mu\}]$ is the coefficient of $\{\lambda\}$ in $\{m\}\otimes\{\mu\}$. This D_λ notation will be used throughout the paper. An explicit result for each of

$$D_\lambda\{m\}^4, \quad D_\lambda\{m\}^2\{m\}^{(2)}, \quad D_\lambda\{m\}\{m\}^{(3)}, \quad D_\lambda\{m\}^{(2)}\{m\}^{(2)} \quad \text{and} \quad D_\lambda\{m\}^{(4)}$$

is obtained for every λ and m, and so $D_\lambda[\{m\}\otimes\{\mu\}]$ can be obtained by substitution in the above expression for $\{m\}\otimes\{\mu\}$.

3. Evaluation of $D_\lambda\{m\}^n$

The following general theorem was given by Duncan (1952*a*) and is useful in obtaining the coefficients of all the S-functions without zero parts in $\{m+1\}^n$ when the full expansion of $\{m\}^n$ is known.

Theorem 1. *If* $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *is any partition of* mn, *then*

$$D_{\lambda_1+1, \lambda_2+1, \ldots, \lambda_n+1}\{m+1\}^n = D_\lambda\{m\}^n.$$

Proof. By the Littlewood-Richardson rule for the multiplication of S-functions, the number of ways of building the Young diagram (λ) from $m\ \alpha_1$'s, $m\ \alpha_2$'s, ..., $m\ \alpha_n$'s is the same as the number of ways the diagram $(\lambda_1+1, \lambda_2+1, \ldots, \lambda_n+1)$ can be built up from $m+1\ \alpha_1$'s, $m+1\ \alpha_2$'s, ..., $m+1\ \alpha_n$'s, since the diagrams differ merely by a column consisting of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

The following is another general theorem, giving a specific formula for the coefficient of $\{\lambda\}$ in $\{m\}^n$ in all cases in which $\lambda_2 \leqslant m$:

Theorem 2. *If* $\lambda_2 \leqslant m$, *then*

$$D_\lambda\{m\}^n = \frac{\prod_{r<s}(\lambda_r - \lambda_s - r + s)}{(n-2)!\,(n-3)!\,(n-4)!\ldots 1},$$

where r, s *take all values from* 2 *to* n.

Proof. In building the diagram (λ) from the mn symbols $\alpha_i, \beta_i, \ldots$, where $i = 1, 2, \ldots, m$, the first row must necessarily start with $\alpha_1, \alpha_2, \ldots, \alpha_m$. The number of ways of completing the diagram is now the number of ways of building the diagram $(\lambda_2, \lambda_3, \ldots, \lambda_n)$ with the β_i's, γ_i's, ..., where not all the symbols need be used. Any $\beta_i, \gamma_i, \ldots$ not used in $(\lambda_2, \lambda_3, \ldots, \lambda_n)$ are merely added to the right of the m α_i's in the first row of (λ) in order to give ultimately λ_1 symbols in the first row.

But if $\{\eta\} = \Sigma K_{\eta\nu} x_1^{\nu_1} x_2^{\nu_2} \ldots x_{n-1}^{\nu_{n-1}}$, then $K_{\eta\nu}$ is the number of standard Young tableaux that can be formed from the symbol x_1 used ν_1 times, the symbol x_2 used ν_2 times, and so on, no symbol being repeated in the same column (Littlewood 1950, p. 191). Hence, taking

$$\{\eta\} = \{\lambda_2, \lambda_3, \ldots, \lambda_n\},$$

the number of ways of building (η), and consequently (λ), is the sum of the numerical coefficients $K_{\eta\nu}$. This sum is obtained by putting $x_1 = x_2 = \ldots = x_n = 1$. An expression for the numerical value of an S-function when each of the quantities involved is unity has been given by Littlewood (1950, p. 126) in the form

$$\{\eta\} = \{\eta_1 \eta_2 \eta_3 \ldots\} = \chi_0^{(\eta)} Q_\eta / q!,$$

where q is the number of which (η) is a partition, Q_η is the product of the first η_i terms from each ith row of

$$\begin{array}{l} n-1, n, n+1, \ldots, \\ n-2, n-1, n, n+1, \ldots, \\ n-3, n-2, n-1, n, \ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \end{array}$$

and $\chi_0^{(\eta)}$ has its usual significance as the group characteristic of the class (1^q) in the representation of the symmetric group which is associated with (η). But $q = \lambda_2 + \lambda_3 + \ldots + \lambda_n$, and

$$\chi_0^{(\eta)} = \frac{q! \prod_{r<s} (\lambda_r - \lambda_s - r + s)}{\Pi[\lambda_r + (n-1) - (r-1)]!}$$

for $r, s = 2, 3, \ldots, n$. Also

$$\begin{aligned} Q_\eta = {} & (n-1)\, n\, (n+1) \ldots [(n-1) + \lambda_2 - 1] \\ & \times (n-2)\,(n-1)\, n \ldots [(n-2) + \lambda_3 - 1] \\ & \times \ldots\ldots\ldots\ldots\ldots\ldots \\ & \times 1.2.3 \ldots\ldots \lambda_n, \end{aligned}$$

and so the theorem follows.

Corollary 1. *If*

$$\{\lambda\}_\alpha = \{\lambda_1 - (n-1)\,\alpha, \lambda_2 + \alpha, \lambda_3 + \alpha, \ldots, \lambda_n + \alpha\},$$

where $\lambda_2 \leqslant m$ *and* $m - \lambda_2 \geqslant \alpha \geqslant -\lambda_n$, *then* $D_{\lambda_\alpha}\{m\}^n = D_\lambda\{m\}^n$.

Corollary 2. *If*

$$\{\lambda\}'_\beta = \{2nm - (n-1)\,\beta - \lambda_1, \beta - \lambda_n, \beta - \lambda_{n-1}, \ldots, \beta - \lambda_2\},$$

where $\lambda_2 \leqslant m$ *and* $m + \lambda_n \geqslant \beta \geqslant \lambda_2$, *then* $D_{\lambda'_\beta}\{m\}^n = D_\lambda\{m\}^n$.

Corollary 3. *If* $\lambda_2 \leqslant m$, *then* $D_\lambda\{m\}^3 = \lambda_2 - \lambda_3 + 1$. *This result has been given by Thrall* (1942).

Corollary 4. *If* $\lambda_2 \leqslant m$, *then* $D_\lambda\{m\}^4 = \frac{1}{2}(\lambda_2 - \lambda_3 + 1)\,(\lambda_2 - \lambda_4 + 2)\,(\lambda_3 - \lambda_4 + 1)$.

Corollary 5. *The coefficient given by the theorem is never zero, since* $\lambda_r - r \neq \lambda_s - s$.

Corollary 6. *If* $\lambda_2 \leqslant m$, *then* $D_{n+\lambda_1, \lambda_2, \ldots, \lambda_n}\{m+1\}^n = D_\lambda\{m\}^n$.

It has been pointed out to me by Mr W. R. Perkins that an expression for the coefficient of $\{\lambda\}$ in $\{m\}^n$, when $\lambda_2 \leqslant m$, has been given by Kostka (1883). This expression, while it can be shown to be equivalent to that in the above theorem, is expressed in terms of the parts of the conjugate partition $(\tilde{\lambda})$ and is not in as simple a form as the above. Its derivation also is much more involved.

THEOREM 3. *If* $\lambda_1 \leqslant 2m$ *and* $\{2m-\bar{\lambda}\}$ *denotes* $\{2m-\lambda_n, 2m-\lambda_{n-1}, 2m-\lambda_{n-2}, \ldots, 2m-\lambda_1\}$, *then* $D_{2m-\bar{\lambda}}\{m\}^n = D_\lambda\{m\}^n$.

Proof. It has to be shown that with each method of constructing a Young tableau in the shape of the partition graph of (λ) from n sets $\alpha_i, \beta_i, \ldots$, each of m symbols, there is associated a unique method of constructing a tableau corresponding to $(2m-\bar{\lambda})$ from n sets $\alpha_i', \beta_i', \ldots$, each of m symbols.

Consider a rectangle of n rows, each of $2m$ nodes. Let a Young tableau corresponding to (λ) be constructed with the mn symbols $\alpha_i, \beta_i, \ldots$, the tableau occupying the top left-hand corner of the rectangle. The first row will have $\alpha_1, \alpha_2, \ldots, \alpha_m$ in the first m places. Place $\alpha_1', \alpha_2', \ldots, \alpha_m'$ in the last m places in the last row of the rectangle. There will thus be an α_i or an α_i' in every column of the rectangle. There will be $m\ \beta_i$'s shared between the first two rows of the rectangle. Place $\beta_1', \beta_2', \ldots, \beta_m'$ in the last two rows of the rectangle so that every column has either a β_i or a β_i'. Similarly, place $\gamma_1', \gamma_2', \ldots, \gamma_m'$ in the last three rows of the rectangle such that every column has either a γ_i or a γ_i'. This procedure is continued until the rectangle is completely filled with the symbols. It is evident that the bottom right-hand corner of the rectangle will contain an inverted Young tableau corresponding to $(2m-\bar{\lambda})$. The theorem follows.

COROLLARY 1. *If* $\lambda_2 \leqslant m$, *and* $m-\lambda_2 \geqslant \alpha \geqslant -\lambda_n$, *and* $\lambda_1-(n-1)\,\alpha \leqslant 2m$, *then*

$$D_\lambda\{m\}^n = D_{2m-\bar{\lambda}_\alpha}\{m\}^n.$$

COROLLARY 2. *If* $\lambda_2 \leqslant m$, *and* $m+\lambda_n \geqslant \beta \geqslant \lambda_2$, *and* $\lambda_1 \geqslant (2m-\beta)\,(n-1)$, *then*

$$D_\lambda\{m\}^n = D_{2m-\bar{\lambda}'_\beta}\{m\}^n.$$

COROLLARY 3. *If* $\beta = m$, *the last result gives, when* $\lambda_2+\lambda_3+\ldots+\lambda_n \leqslant m$,

$$D_{m+\lambda_2,\, m+\lambda_3, \ldots,\, m+\lambda_n,\, m-(\lambda_2+\lambda_3+\ldots+\lambda_n)}\{m\}^n = D_\lambda\{m\}^n.$$

Theorems 2 and 3 and their corollaries do not cover all the S-functions appearing in $\{m\}^n$, but they do make possible some grouping of these S-functions. Thus theorem 2 determines that the coefficient of $\{27.1\}$ in $\{7\}^4$ is 3. The first corollary to this theorem gives $6 \geqslant \alpha \geqslant 0$, and so the coefficients of $\{27.1\}_1, \{27.1\}_2, \ldots, \{27.1\}_6$ are also 3. These S-functions are respectively $\{24.21^2\}$, $\{21.32^2\}$, $\{18.43^2\}$, $\{15.54^2\}$, $\{12.65^2\}$ and $\{976^2\}$. The second corollary gives $7 \geqslant \beta \geqslant 1$ and leads to the seven S-functions $\{27.1\}_1', \{27.1\}_2', \ldots, \{27.1\}_7'$, each of which has coefficient 3. These are respectively $\{26.1^2\}$, $\{23.2^21\}$, $\{20.3^22\}$, $\{17.4^23\}$, $\{14.5^24\}$, $\{11.6^25\}$ and $\{87^26\}$. Theorem 3 can now be applied to $\{27.1\}_5$, $\{27.1\}_6$, $\{27.1\}_5'$, $\{27.1\}_6'$, giving a further four S-functions

$$\{14-\bar{\lambda}_5\} = \{9^282\}, \quad \{14-\bar{\lambda}_6\} = \{8^275\}, \quad \{14-\bar{\lambda}_5'\} = \{10.9^2\} \quad \text{and} \quad \{14-\bar{\lambda}_6'\} = \{98^23\},$$

which also have 3 as coefficient in $\{7\}^4$. It so happens in this example that all S-functions with coefficient 3 have been given above, but this need not occur in every case. That this set of eighteen S-functions may be defined by any one of them is implied by the following theorem:

THEOREM 4.

(i) $\{\{\lambda\}_{\alpha_1}\}_{\alpha_2}=\{\lambda\}_{\alpha_1+\alpha_2}$, (ii) $\{\{\lambda\}'_{\beta_1}\}'_{\beta_2}=\{\lambda\}_{\beta_2-\beta_1}$,

(iii) $\{\{\lambda\}_{\alpha}\}'_{\beta}=\{\lambda\}'_{\beta-\alpha}$, (iv) $\{\{\lambda\}'_{\beta}\}_{\alpha}=\{\lambda\}'_{\alpha+\beta}$.

Proof. (i) $\{\lambda\}_{\alpha_1}=\{\lambda_1-(n-1)\,\alpha_1,\lambda_2+\alpha_1,\lambda_3+\alpha_1,\ldots,\lambda_n+\alpha_1\}$, where $m-\lambda_2\geqslant\alpha_1\geqslant-\lambda_n$ and $\lambda_2\leqslant m$. Also

$$\{\{\lambda\}_{\alpha_1}\}_{\alpha_2}=\{\lambda_1-(n-1)\,(\alpha_1+\alpha_2),\lambda_2+\alpha_1+\alpha_2,\lambda_3+\alpha_1+\alpha_2,\ldots,\lambda_n+\alpha_1+\alpha_2\}=\{\lambda\}_{\alpha_1+\alpha_2},$$

where $m-\lambda_2-\alpha_1\geqslant\alpha_2\geqslant-\lambda_n-\alpha_1$, which is $m-\lambda_2\geqslant\alpha_1+\alpha_2\geqslant-\lambda_n$.

(ii) $\{\lambda\}'_{\beta_1}=\{2nm-(n-1)\,\beta_1-\lambda_1,\beta_1-\lambda_n,\beta_1-\lambda_{n-1},\ldots,\beta_1-\lambda_2\}$, where $\lambda_2\leqslant m$ and

$$m+\lambda_n\geqslant\beta_1\geqslant\lambda_2.$$

$$\{\{\lambda\}'_{\beta_1}\}'_{\beta_2}=\{2nm-(n-1)\,\beta_2-[2nm-(n-1)\,\beta_1-\lambda_1],\beta_2-\beta_1+\lambda_2,\beta_2-\beta_1+\lambda_3,\ldots,\beta_2-\beta_1+\lambda_n\},$$

where $m+\beta_1-\lambda_2\geqslant\beta_2\geqslant\beta_1-\lambda_n$, giving $m-\lambda_2\geqslant\beta_2-\beta_1\geqslant-\lambda_n$. Hence

$$\{\{\lambda\}'_{\beta_1}\}'_{\beta_2}=\{\lambda_1-(n-1)\,(\beta_2-\beta_1),\lambda_2+(\beta_2-\beta_1),\lambda_3+(\beta_2-\beta_1),\ldots,\lambda_n+(\beta_2-\beta_1)\}=\{\lambda\}_{\beta_2-\beta_1}.$$

(iii) $\{\{\lambda\}_{\alpha}\}'_{\beta}=\{2nm-(n-1)\,\beta-\lambda_1+(n-1)\,\alpha,\beta-\lambda_n-\alpha,\beta-\lambda_{n-1}-\alpha,\ldots,\beta-\lambda_2-\alpha\}$, where $m+\lambda_n+\alpha\geqslant\beta\geqslant\lambda_2+\alpha$. Hence $\{\{\lambda\}_{\alpha}\}'_{\beta}=\{\lambda\}'_{\beta-\alpha}$, where $m+\lambda_n\geqslant\beta-\alpha\geqslant\lambda_2$.

(iv) $\{\{\lambda\}'_{\beta}\}_{\alpha}=\{2nm-(n-1)\,(\beta+\alpha)-\lambda_1,\beta+\alpha-\lambda_n,\beta+\alpha-\lambda_{n-1},\ldots,\beta+\alpha-\lambda_2\}=\{\lambda\}'_{\alpha+\beta}$,

where $m+\lambda_n\geqslant\alpha+\beta\geqslant\lambda_2$.

Results additional to theorems 2 and 3 would be needed to give the coefficients of the remaining types of S-function in the general case $\{m\}^n$, but for $n=4$ the following theorem covers all the outstanding cases:

THEOREM 5. *If* $\{\lambda\}=\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$, *where* $\lambda_2=m+k$, $\lambda_3=m-l$, *and* $m\geqslant l$, $k\geqslant 0$, *then*

(i) *if* $k\leqslant l+1$,

$$D_{\lambda}\{m\}^4=\tfrac{1}{2}(\lambda_3-\lambda_4+1)\,(\lambda_2-\lambda_3+1)\,(\lambda_2-\lambda_4+2)-2k(k+1)\,(\lambda_3-\lambda_4+1);$$

(ii) *if* $k>l+1$,

$$D_{\lambda}\{m\}^4=\tfrac{1}{2}(\lambda_3-\lambda_4+1)\,(\lambda_2-\lambda_3+1)\,(\lambda_2-\lambda_4+2)-2k(k+1)\,(\lambda_3-\lambda_4+1)+(k-l-1)\,(k-l)\,(k-l+1).$$

Proof. The coefficient of $\{\xi_1,\xi_2,\xi_3\}$ in $\{m\}^3$ has been shown by Thrall (1942) to be

$$\min\,(1+\xi_1-\xi_2,1+\xi_2-\xi_3)$$

or equivalently, is $1+\xi_1-\xi_2$ if $\xi_2\geqslant m$, and is $1+\xi_2-\xi_3$ if $\xi_2\leqslant m$. Let $\{\xi_1,\xi_2,\xi_3\}$ be an S-function such that the product $\{\xi_1,\xi_2,\xi_3\}\{m\}$ contains $\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$. Then

$$2m-k+l-\lambda_4\geqslant\xi_1\geqslant m+k\geqslant\xi_2\geqslant m-l\geqslant\xi_3\geqslant\lambda_4.$$

(*a*) Take $\xi_2=m-l+r$, where $r=0,1,2,\ldots,l$. Then $\xi_1+\xi_3=2m+l-r$, and the least value of ξ_3 is $\max\,(2m+l-r-2m+k-l+\lambda_4,\lambda_4)$, which may be written as $\lambda_4+((k-r))$, where the double brackets mean that the number contained by them is to be taken as zero when it is negative. Similarly, the greatest value of ξ_3 is $\lambda_3-((k-2l+r))$.

Using Thrall's result, the sum of the coefficients of the S-functions $\{\xi_1, \xi_2, \xi_3\}$ which contribute to $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and which have $\xi_2 \leqslant m$ is the sum of the consecutive integers from $1+(\lambda_3+r)-[\lambda_3-((k-2l+r))]$ to $1+(\lambda_3+r)-[\lambda_4+((k-r))]$ inclusive, this sum being then summed from $r = 0$ to $r = l$. For a fixed r, the sum of the consecutive integers is

$$\tfrac{1}{2}[\lambda_3-\lambda_4+r+1-((k-r))]\,[\lambda_3-\lambda_4+r+2-((k-r))] \\ -\tfrac{1}{2}[r+((k-2l+r))]\,[r+1+((k-2l+r))].$$

The sum of these expressions from $r = 0$ to $r = l$ depends on whether k is greater or less than l. When $k \leqslant l$, the sum is

$$\frac{1}{2}\sum_{r=0}^{l}[(\lambda_3-\lambda_4+1)^2+(\lambda_3-\lambda_4+1)\,(2r+1)]-(\lambda_3-\lambda_4+1)\sum_{r=0}^{k}(k-r) \\ +\frac{1}{2}\sum_{r=0}^{k}[(k-r)^2-(k-r)\,(2r+1)]$$
$$= \tfrac{1}{2}(\lambda_3-\lambda_4+1)^2\,(l+1)+\tfrac{1}{2}(\lambda_3-\lambda_4+1)\,(l+1)^2-\tfrac{1}{2}(\lambda_3-\lambda_4+1)\,k(k+1)$$
$$= \tfrac{1}{2}(\lambda_3-\lambda_4+1)\,[(\lambda_3-\lambda_4+l+2)\,(l+1)-k(k+1)]. \tag{1}$$

When $k \geqslant l$, the sum is

$$\frac{1}{2}\sum_{r=0}^{l}[(\lambda_3-\lambda_4+1)^2+(\lambda_3-\lambda_4+1)\,(2r+1)]-(\lambda_3-\lambda_4+1)\sum_{r=0}(k-r) \\ +\frac{1}{2}\sum_{r=0}^{l}[(k-r)^2-(k-r)\,(2r+1)]-\frac{1}{2}\sum_{r=2l-k}^{l}[(k-2l+r)^2+(k-2l+r)\,(2r+1)]$$
$$= \tfrac{1}{2}(\lambda_3-\lambda_4+1)\,(\lambda_3-\lambda_4+l+2)\,(l+1)-\tfrac{1}{2}(\lambda_3-\lambda_4+1)\,(l+1)\,(2k-l) \\ +\tfrac{1}{2}(k-l)\,(k-l-1)\,(l+1)-\tfrac{1}{2}(k-l)\,(k-l+1)\,(l+1)$$
$$= \tfrac{1}{2}(l+1)\,[(\lambda_3-\lambda_4+1)-2(k-l)]\,(\lambda_3-\lambda_4+2)$$
$$= \tfrac{1}{2}(l+1)\,(\lambda_1-\lambda_2+1)\,(\lambda_3-\lambda_4+2). \tag{2}$$

(*b*) Take $\xi_2 = m+s+1$, where $s = 0, 1, 2, \ldots, k-1$. Then $\xi_1+\xi_3 = 2m-s-1$, and the least value of ξ_1 is $\max\,[m+k, 2m-s-1-(m-l)]$, which can be written as $\lambda_2+((l-k-s-1))$. Similarly, the greatest value of ξ_1 is $\lambda_1-((l-k+s+1))$. Since $\xi_2 > m$, the alternative form of Thrall's lemma is applied to every $\{\xi_1, \xi_2, \xi_3\}$ with $\xi_2 = m+s+1$, giving the sum, for a given s, of the consecutive integers from

$$1+\lambda_2+((l-k-s-1))-(m+s+1) \quad \text{to} \quad 1+\lambda_1-((l-k+s+1))-(m+s+1)$$

inclusive. This sum is

$$\tfrac{1}{2}[\lambda_1-m-s-((l-k+s+1))]\,[\lambda_1-m-s+1-((l-k+s+1))] \\ -\tfrac{1}{2}[k-s-1+((l-k-s-1))]\,[k-s+((l-k-s-1))].$$

The summation of this expression from $s = 0$ to $k-1$ again depends on the relative magnitudes of k and l.

When $k \geqslant l$, the sum is

$$\frac{1}{2}\sum_{s=0}^{k-1}(\lambda_1-m-s)\,(\lambda_1-m-s+1)-\frac{1}{2}\sum_{s=0}^{k-1}(k-s-1)\,(k-s) \\ -\frac{1}{2}\sum_{s=k-l-1}^{k-1}(2\lambda_1-2m-2s+1)\,(l-k+s+1)+\frac{1}{2}\sum_{s=k-l-1}^{k-1}(l-k+s+1)^2,$$

which reduces to

$$\tfrac{1}{2}k(\lambda_1-m+1)(\lambda_1-m+1-k)-\tfrac{1}{2}l(l+1)(\lambda_1-m)+\tfrac{1}{2}l(l+1)(k-1)$$
$$=\tfrac{1}{2}(\lambda_1-\lambda_2+1)[(\lambda_1-\lambda_2+k+1)k-l(l+1)]. \qquad (3)$$

When $k<l$, the sum is

$$\frac{1}{2}\sum_{s=0}^{k-1}(\lambda_1-m-s)(\lambda_1-m-s+1)-\frac{1}{2}\sum_{s=0}^{k-1}(k-s-1)(k-s)$$
$$-\frac{1}{2}\sum_{s=0}^{k-1}(2\lambda_1-2m-2s+1)(l-k+s+1)+\frac{1}{2}\sum_{s=0}^{k-1}(l-k+s+1)^2$$
$$-\frac{1}{2}\sum_{s=0}^{l-k-1}(l-k-s-1)(2k-2s-1)-\frac{1}{2}\sum_{s=0}^{l-k-1}(l-k-s-1)^2$$
$$=\tfrac{1}{2}k(\lambda_1-m+1)(\lambda_1-m+1-k)-\tfrac{1}{2}k(\lambda_1-m+1)(2l-k+1)+\tfrac{1}{2}k(k+1)(2l-k)$$
$$=\tfrac{1}{2}k(\lambda_1-\lambda_2+1)(\lambda_1-\lambda_2+1+k)-\tfrac{1}{2}k(\lambda_1-\lambda_2+1)(2l-k+1)+k(l-k)$$
$$=\tfrac{1}{2}k(\lambda_1-\lambda_2+1-2l+2k)(\lambda_1-\lambda_2)$$
$$=\tfrac{1}{2}k(\lambda_3-\lambda_4+1)(\lambda_1-\lambda_2). \qquad (4)$$

This result is clearly valid for $k=l$, since substituting $k=l$ in (3) gives $\tfrac{1}{2}l(\lambda_1-\lambda_2+1)(\lambda_1-\lambda_2)$ which agrees with (4) when $k=l$.

Hence, combining (1) and (4), the coefficient of $\{\lambda\}$ in $\{m\}^4$, when $k\leqslant l$, is

$$\tfrac{1}{2}(\lambda_3-\lambda_4+1)[(m-\lambda_4+2)(m-\lambda_3+1)-k(k+1)+k(\lambda_1-\lambda_2)]$$
$$=\tfrac{1}{2}(\lambda_3-\lambda_4+1)[(\lambda_2-\lambda_4+2-k)(\lambda_2-\lambda_3+1-k)-k(k+1)+k(\lambda_1-\lambda_2)]$$
$$=\tfrac{1}{2}(\lambda_3-\lambda_4+1)[(\lambda_2-\lambda_4+2)(\lambda_2-\lambda_3+1)-k(2\lambda_2-\lambda_3-\lambda_4+3)-k+k(\lambda_1-\lambda_2)]$$
$$=\tfrac{1}{2}(\lambda_3-\lambda_4+1)[(\lambda_2-\lambda_4+2)(\lambda_2-\lambda_3+1)-4k(k+1)]. \qquad (5)$$

Similarly, combining (2) and (3), the coefficient of $\{\lambda\}$, when $k\geqslant l$, is

$$\tfrac{1}{2}(\lambda_1-\lambda_2+1)[(\lambda_3-\lambda_4+1)(l+1)+1-l^2+k^2+k(\lambda_1-\lambda_2+1)]$$
$$=\tfrac{1}{2}(\lambda_1-\lambda_2+1)[(\lambda_3-\lambda_4+1)(l+1)+1-l^2+k^2+k(\lambda_3-\lambda_4+1)+2k(l-k)]$$
$$=\tfrac{1}{2}(\lambda_1-\lambda_2+1)[(\lambda_3-\lambda_4+1)(\lambda_2-\lambda_3+1)+1-(l-k)^2]$$
$$=\tfrac{1}{2}[(\lambda_2-\lambda_4+2)+(l-3k-1)][(\lambda_3-\lambda_4+1)(\lambda_2-\lambda_3+1)+1-(l-k)^2]$$
$$=\tfrac{1}{2}(\lambda_2-\lambda_4+2)(\lambda_3-\lambda_4+1)(\lambda_2-\lambda_3+1)+\tfrac{1}{2}(\lambda_3-\lambda_4+1)(l-3k-1)(l+k+1)$$
$$+\tfrac{1}{2}(\lambda_1-\lambda_2+1)[1-(l-k)^2]$$
$$=\tfrac{1}{2}(\lambda_2-\lambda_4+2)(\lambda_3-\lambda_4+1)(\lambda_2-\lambda_3+1)+\tfrac{1}{2}(\lambda_3-\lambda_4+1)(l^2-2kl-3k^2-4k-1)$$
$$+\tfrac{1}{2}[(\lambda_3-\lambda_4+1)+2(l-k)][1-(l-k)^2]$$
$$=\tfrac{1}{2}(\lambda_2-\lambda_4+2)(\lambda_3-\lambda_4+1)(\lambda_2-\lambda_3+1)-2(\lambda_3-\lambda_4+1)k(k+1)$$
$$+(k-l-1)(k-l)(k-l+1). \qquad (6)$$

Since (5) and (6) differ only in respect of the extra product $(k-l-1)(k-l)(k-l+1)$, (5) will hold for $k\leqslant l+1$, and (6) for $k>l+1$. This completes the proof of the theorem.

It is of some interest to conjecture general results of which those given above are particular cases. Examination of a number of numerical cases suggests that for $n=5$,

(i) if $\{\lambda\}=\{\lambda_1, m+k, m-l, \lambda_4, \lambda_5\}$, where $m\geqslant k$, $l\geqslant 0$ and $k\leqslant l$, then

$$D_\lambda\{m\}^5=\tfrac{1}{12}(\lambda_2-\lambda_3+1)(\lambda_2-\lambda_4+2)(\lambda_2-\lambda_5+3)(\lambda_3-\lambda_4+1)(\lambda_3-\lambda_5+2)(\lambda_4-\lambda_5+1)$$
$$-\frac{5k(k+1)(k+2)}{3!}\tfrac{1}{2}(\lambda_3-\lambda_4+1)(\lambda_3-\lambda_5+2)(\lambda_4-\lambda_5+1);$$

(ii) if $\{\lambda\} = \{\lambda_1, m+k, m-l, m-j, \lambda_5\}$, where $k \geq l$, then to the foregoing expression for $D_\lambda\{m\}^5$ must be added

$$\tfrac{1}{12}[(k-l+3)(k-l+2)(k-l+1)(k-l)(k-l-1)(\lambda_4-\lambda_5+1) \\ -(k-j+2)(k-j+1)(k-j)(k-j-1)(k-j-2)(\lambda_3-\lambda_5+2) \\ -(k-l-j+2)(k-l-j+1)(k-l-j)(k-l-j-1)(k-l-j-2)(\lambda_3-\lambda_4+1)],$$

where any negative terms in the products are regarded as zeros.

The generalization of (i) would appear to be that if $\{\lambda\} = \{\lambda_1, m+k, m-l, \lambda_4, \ldots, \lambda_n\}$, where $m \geq k$, $l \geq 0$ and $k \leq l$, then

$$D_\lambda\{m\}^n = \frac{\prod\limits_{r,s=2}^{n}(\lambda_r-\lambda_s-r+s)}{(n-2)!\,(n-3)!\ldots 1!} - \frac{n}{(n-2)!}k(k+1)\ldots(k+n-3)\frac{\prod\limits_{r,s=3}^{n}(\lambda_r-\lambda_s-r+s)}{(n-3)!\,(n-4)!\ldots 1!},$$

where $r<s$.

A complete generalization of (ii) would probably be complicated. No proofs of these conjectures have as yet been obtained, but they have been found to give correct results in a large number of numerical cases, and it appears likely that such generalizations of theorem 5 do exist.

4. Evaluation of $D_\lambda\{m\}^2\{m\}^{(2)}$

Theorem 6. *If* $(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *is a partition of* $4m$, *then* $D_\lambda\{m\}^2\{m\}^{(2)} = \pm(N_e - N_o)$, *where* N_e *and* N_o *are the numbers of even and odd integers respectively in the ranges specified by table* 1, *and the sign is given by the last column of the table. For all S-functions* $\{\lambda\}$ *not specified in the table the result is zero.*

Table 1

$\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4$	range for even p: minimum of	range for even p: maximum of	range for odd p: minimum of	range for odd p: maximum of	sign
e e e e	$2m-\lambda_3, \lambda_2, m$	$2m-\lambda_1, \lambda_4$	m	$2m+1-\lambda_2, \lambda_3+1$	+
o o o o	m	$2m+1-\lambda_2, \lambda_3+1$	$2m-\lambda_3, \lambda_2, m$	$2m-\lambda_1, \lambda_4$	+
o o e e	$2m-\lambda_3, m$	$2m+1-\lambda_2, \lambda_4$	λ_2, m	$2m-\lambda_1, \lambda_3+1$	−
e e o o	λ_2, m	$2m-\lambda_1, \lambda_3+1$	$2m-\lambda_3, m$	$2m+1-\lambda_2, \lambda_4$	−
o e e o	m	$2m+2-\lambda_3, \lambda_2+2$	$2m-\lambda_2-1, \lambda_3-1, m$	$2m-\lambda_1, \lambda_4$	−
e o o e	$2m-\lambda_2-1, \lambda_3-1, m$	$2m-\lambda_1, \lambda_4$	m	$2m+2-\lambda_3, \lambda_2+2$	−

Proof. Since $\{m\}^2 = \{2m\}+\{2m-1,1\}+\{2m-2,2\}+\ldots+\{m^2\}$, then

$$D_\lambda\{m\}^2\{m\}^{(2)} = \sum_p \begin{vmatrix} D_{\lambda_1-2m+p} & D_{\lambda_1+1-p} & D_{\lambda_1+2} & D_{\lambda_1+3} \\ D_{\lambda_2-1-2m+p} & D_{\lambda_2-p} & D_{\lambda_2+1} & D_{\lambda_2+2} \\ D_{\lambda_3-2-2m+p} & D_{\lambda_3-1-p} & D_{\lambda_3} & D_{\lambda_3+1} \\ D_{\lambda_4-3-2m+p} & D_{\lambda_4-2-p} & D_{\lambda_4-1} & D_{\lambda_4} \end{vmatrix}\{m\}^{(2)}$$

$$= \sum_p \Delta_p\{m\}^{(2)},$$

where the range for p giving non-zero Δ_p is $\max(0, 2m-\lambda_1) \leq p \leq \min(\lambda_2, m)$.

The next step in the procedure (Foulkes 1949) is to divide every even suffix in each Δ_p by 2 and to regard every D with an odd suffix as a zero element of the determinant. The result now depends on the parities of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and p.

(i) $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ *all even*

If p is even,

$$\Delta_p = \begin{vmatrix} D^{(2)}_{\frac{1}{2}(\lambda_1-2m+p)} & \cdot & D^{(2)}_{\frac{1}{2}(\lambda_1+2)} & \cdot \\ \cdot & D^{(2)}_{\frac{1}{2}(\lambda_2-p)} & \cdot & D^{(2)}_{\frac{1}{2}(\lambda_2+2)} \\ D^{(2)}_{\frac{1}{2}(\lambda_3-2-2m+p)} & \cdot & D^{(2)}_{\frac{1}{2}\lambda_3} & \cdot \\ \cdot & D^{(2)}_{\frac{1}{2}(\lambda_4-2-p)} & \cdot & D^{(2)}_{\frac{1}{2}\lambda_4} \end{vmatrix}.$$

This determinant operating on $\{m\}^{(2)}$ will give a zero result if every suffix is positive or zero, since each operator may be replaced by a unit as far as the final result is concerned. It will give $+1$ if, and only if, all the conditions

$$\lambda_3-2-2m+p<0, \quad \lambda_4-2-p<0, \quad \lambda_1-2m+p\geqslant 0, \quad \lambda_2-p\geqslant 0$$

are satisfied, that is, when

$$\min(2m+1-\lambda_3, \lambda_2, m)\geqslant p\geqslant \max(2m-\lambda_1, \lambda_4-1, 0),$$

which is equivalent to

$$\min(2m-\lambda_3, \lambda_2, m)\geqslant p\geqslant \max(2m-\lambda_1, \lambda_4), \tag{7}$$

since p is even.

The determinant Δ_p can never give rise to -1 when p is even. Hence every even value of p in the above range gives a Δ_p which leads to $+1$ in $D_\lambda\{m\}^2\{m\}^{(2)}$.

When p is odd,

$$\Delta_p = \begin{vmatrix} \cdot & D^{(2)}_{\frac{1}{2}(\lambda_1+1-p)} & D^{(2)}_{\frac{1}{2}(\lambda_1+2)} & \cdot \\ D^{(2)}_{\frac{1}{2}(\lambda_2-1-2m+p)} & \cdot & \cdot & D^{(2)}_{\frac{1}{2}(\lambda_2+2)} \\ \cdot & D^{(2)}_{\frac{1}{2}(\lambda_3-1-p)} & D^{(2)}_{\frac{1}{2}\lambda_3} & \cdot \\ D^{(2)}_{\frac{1}{2}(\lambda_4-3-2m+p)} & \cdot & \cdot & D^{(2)}_{\frac{1}{2}\lambda_4} \end{vmatrix}.$$

This determinant operating on $\{m\}^{(2)}$ will give a zero result unless

$$\lambda_1+1-p\geqslant 0, \quad \lambda_2-1-2m+p\geqslant 0, \quad \lambda_3-1-p<0, \quad \lambda_4-3-2m+p<0,$$

in which case it will give -1. Hence every odd value of p in the range

$$\min(2m+2-\lambda_4, \lambda_1+1, m)\geqslant p\geqslant \max(2m+1-\lambda_2, \lambda_3, 0)$$

gives -1 in $D_\lambda\{m\}^2\{m\}^{(2)}$. This range reduces to

$$m\geqslant p\geqslant \max(2m+1-\lambda_2, \lambda_3+1). \tag{8}$$

Hence when $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are all even, $D_\lambda\{m\}^2\{m\}^{(2)} = N_e-N_o$, where N_e, N_o are respectively the number of even integers, including zero, satisfying (7), and the number of odd integers satisfying (8).

(ii) $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ *all odd*

An analogous argument leads to the result that in this case $D_\lambda\{m\}^2\{m\}^{(2)} = N_e-N_o$, where N_e is the number of even integers satisfying (8), and N_o is the number of odd integers satisfying (7). Repetition of the argument for all the possible cases of odd and even λ_i gives the required result.

As a simple example take $m = 6$ and $\{\lambda\} = \{12.75\}$. The last row of table 1 gives

$$\min(4, 4, 6)\geqslant p_e\geqslant \max(0, 0) \quad \text{and} \quad 6\geqslant p_o\geqslant \max(9, 9),$$

so that $N_e = 3$ and $N_o = 0$, giving the coefficient of $\{12.75\}$ in $\{6\}^2\{6\}^{(2)}$ as -3.

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THEOREM 7. $D_{\lambda_1+1,\lambda_2+1,\lambda_3+1,\lambda_4+1}\{m+1\}^2\{m+1\}^{(2)} = -D_\lambda\{m\}^2\{m\}^{(2)}$.

Proof. The addition of a unit to each part of $\{\lambda\}$ changes the parity of each part. Table 1 shows that this implies that the ranges for odd and even p are interchanged. Further, the upper and lower bounds are in all cases increased by unity when λ_i+1 and $m+1$ are written instead of λ_i and m respectively. Hence N_e-N_o for

$$\{\lambda_1+1, \lambda_2+1, \lambda_3+1, \lambda_4+1\}$$

in $\{m+1\}^2\{m+1\}^{(2)}$ is that of $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ in $\{m\}^2\{m\}^{(2)}$ with sign reversed.

THEOREM 8. *If* $\lambda_1 \leqslant 2m$, *and*

$$\{2m-\bar{\lambda}\} = \{2m-\lambda_4, 2m-\lambda_3, 2m-\lambda_2, 2m-\lambda_1\}, \quad \textit{then} \quad D_{2m-\bar{\lambda}}\{m\}^2\{m\}^{(2)} = D_\lambda\{m\}^2\{m\}^{(2)}.$$

Proof. The parities of the four parts of $\{2m-\bar{\lambda}\}$ are the same as those of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ in all except the two middle rows of the table, in which they are reversed. Replacing $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ in the table by $2m-\lambda_4$, $2m-\lambda_3$, $2m-\lambda_2$, $2m-\lambda_1$ interchanges the two middle rows but leaves the others unaltered. The theorem follows.

THEOREM 9. *If* $\lambda_2 \leqslant m$, *then*

$$\begin{aligned} D_\lambda\{m\}^2\{m\}^{(2)} &= \tfrac{1}{2}(\lambda_2-\lambda_4+2) \text{ if } \lambda_2, \lambda_3, \lambda_4 \text{ are } e, e, e,\\ &= -\tfrac{1}{2}(\lambda_2-\lambda_4+2) \text{ if } \lambda_2, \lambda_3, \lambda_4 \text{ are } o, o, o,\\ &= \tfrac{1}{2}(\lambda_2-\lambda_3+1) \text{ if } \lambda_2, \lambda_3, \lambda_4 \text{ are } o, e, e,\\ &= -\tfrac{1}{2}(\lambda_2-\lambda_3+1) \text{ if } \lambda_2, \lambda_3, \lambda_4 \text{ are } e, o, o,\\ &= \tfrac{1}{2}(\lambda_3-\lambda_4+1) \text{ if } \lambda_2, \lambda_3, \lambda_4 \text{ are } e, e, o,\\ &= -\tfrac{1}{2}(\lambda_3-\lambda_4+1) \text{ if } \lambda_2, \lambda_3, \lambda_4 \text{ are } o, o, e,\\ &= 0 \text{ in all other cases.} \end{aligned}$$

Proof. Writing the rows in the same order, table 1 now reduces to table 2, from which the theorem is evident.

TABLE 2

range for even p	range for odd p	N_e	N_o	sign
$\lambda_2 \geqslant p \geqslant \lambda_4$	$m \geqslant p \geqslant 2m+1-\lambda_2$	$\frac{1}{2}(\lambda_2-\lambda_4+2)$	0	+
$m \geqslant p \geqslant 2m+1-\lambda_2$	$\lambda_2 \geqslant p \geqslant \lambda_4$	0	$\frac{1}{2}(\lambda_2-\lambda_4+2)$	+
$m \geqslant p \geqslant 2m+1-\lambda_2$	$\lambda_2 \geqslant p \geqslant \lambda_3+1$	0	$\frac{1}{2}(\lambda_2-\lambda_3+1)$	−
$\lambda_2 \geqslant p \geqslant \lambda_3+1$	$m \geqslant p \geqslant 2m+1-\lambda_2$	$\frac{1}{2}(\lambda_2-\lambda_3+1)$	0	−
$m \geqslant p \geqslant 2m+2-\lambda_3$	$\lambda_3-1 \geqslant p \geqslant \lambda_4$	0	$\frac{1}{2}(\lambda_3-\lambda_4+1)$	−
$\lambda_3-1 \geqslant p \geqslant \lambda_4$	$m \geqslant p \geqslant 2m+2-\lambda_3$	$\frac{1}{2}(\lambda_3-\lambda_4+1)$	0	−

COROLLARY 1. *If* $\{\lambda\}_\alpha = \{\lambda_1-3\alpha, \lambda_2+\alpha, \lambda_3+\alpha, \lambda_4+\alpha\}$, *where* $\lambda_2 \leqslant m$ *and* $m-\lambda_2 \geqslant \alpha \geqslant -\lambda_4$, *then* $D_{\lambda_\alpha}\{m\}^2\{m\}^{(2)} = (-1)^\alpha D_\lambda\{m\}^2\{m\}^{(2)}$.

Proof. If α is even, the parities of the four parts of $\{\lambda\}_\alpha$ are the same as those of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ respectively. If α is odd, the parities are reversed. This, with the above theorem, proves the corollary.

COROLLARY 2. *If* $\quad \{\lambda\}'_\beta = \{8m-3\beta-\lambda_1, \beta-\lambda_4, \beta-\lambda_3, \beta-\lambda_2\}$,

where $\lambda_2 \leqslant m$ *and* $m+\lambda_4 \geqslant \beta \geqslant \lambda_2$, *then* $D_{\lambda'_\beta}\{m\}^2\{m\}^{(2)} = (-1)^\beta D_\lambda\{m\}^2\{m\}^{(2)}$.

COROLLARY 3. *If* $\lambda_2 \leqslant m$, $m-\lambda_2 \geqslant \alpha \geqslant -\lambda_4$ *and* $\lambda_1-3\alpha \leqslant 2m$, *then*

$$D_{2m-\bar{\lambda}_\alpha}\{m\}^2\{m\}^{(2)} = (-1)^\alpha D_\lambda\{m\}^2\{m\}^{(2)}.$$

COROLLARY 4. *If* $\lambda_2 \leqslant m$, $m+\lambda_4 \geqslant \beta \geqslant \lambda_2$, *and* $\lambda_1 \geqslant 3(2m-\beta)$, *then*

$$D_{2m-\bar{\lambda}'_\beta}\{m\}^2\{m\}^{(2)} = (-1)^\beta D_\lambda\{m\}^2\{m\}^{(2)}.$$

COROLLARY 5. *If* $\lambda_2 \leqslant m$, *then* $D_{\lambda_1+4,\lambda_2,\lambda_3,\lambda_4}\{m+1\}^2\{m+1\}^{(2)} = D_\lambda\{m\}^2\{m\}^{(2)}$.

5. EVALUATION OF $D_\lambda\{m\}\{m\}^{(n-1)}$

THEOREM 10. $D_\lambda\{m\}\{m\}^{(n-1)} = 1, 0,$ *or* -1.

Proof.

$$D_\lambda\{m\}\{m\}^{(n-1)} = \begin{vmatrix} D_{\lambda_1-m} & D_{\lambda_1+1} & \dots & D_{\lambda_1+n-1} \\ D_{\lambda_2-m-1} & D_{\lambda_2} & \dots & D_{\lambda_2+n-2} \\ \dots & \dots & \dots & \dots \\ D_{\lambda_{n-1}-m-n} & D_{\lambda_{n-1}-n+1} & \dots & D_{\lambda_{n-1}+1} \\ D_{\lambda_n-m-n+1} & D_{\lambda_n-n+2} & \dots & D_{\lambda_n} \end{vmatrix} \{m\}^{(n-1)}.$$

The only D operators which matter are those with suffixes divisible by $n-1$. In each row, one and only one of the suffixes in the last $n-1$ columns is divisible by $n-1$. Several suffixes in the first column may be divisible by $n-1$. Hence $D_\lambda\{m\}\{m\}^{(n-1)}$ is the numerical value of a determinant which may have some units in its first column, the remaining elements in this column being zeros, and the remaining $n-1$ columns are such that they have one unit in each row, the remaining elements in the row being zeros. Such a determinant must have one of the values $1, 0, -1$.

The particular value to be taken when $n = 4$ is given by the following theorem, due to Duncan (1952*a*):

THEOREM 11. $D_\lambda\{m\}\{m\}^{(3)} = 1, 0, -1$ *according as* $D_\lambda\{m\}^4 \equiv 1, 0, -1 \pmod 3$.

Proof. From §2, $D_\lambda[\chi^{(\mu)}_{1^4}\{m\}^4 + 8\chi^{(\mu)}_{13}\{m\}\{m\}^{(3)}] \equiv 0 \pmod 3$. When $(\mu) = (4)$, $\chi^{(4)}_{1^4} = \chi^{(4)}_{13} = 1$, and so $D_\lambda\{m\}^4 \equiv D_\lambda\{m\}\{m\}^{(3)} \pmod 3$.

This is extended to the case when $n-1$ is a prime by the following theorem:

THEOREM 12. *If* $n-1$ *is prime*, $D_\lambda\{m\}^n \equiv D_\lambda\{m\}\{m\}^{(n-1)} \pmod{n-1}$.

Proof. The order of any class $\rho = (1^a 2^b 3^c \dots)$ of the symmetric group of order $n!$ is (Littlewood 1950, p. 40)

$$h_\rho = \frac{n!}{1^a 2^b 3^c \dots a!\,b!\,c!\dots}.$$

When $\rho = (1, n-1)$, then $h_\rho = n(n-2)!$, and since $(n-2)! \equiv -1 \pmod{n-1}$, when $n-1$ is prime, then $h_\rho \equiv -1 \pmod{n-1}$. When ρ is any class other than the identity or $(1, n-1)$, h_ρ will necessarily be divisible by $n-1$, since none of $a, b, c, \dots$ will equal $n-1$ or will contain $n-1$ as a factor. The theorem follows.

It is necessary to obtain some criteria to decide which of the values $-1, 0, 1$ to take in any case not covered by the last theorem. Numerical cases are easily dealt with. Thus, if the coefficient of $\{4^3 21\}$ in $\{3\}\{3\}^{(4)}$ is required, the procedure is as follows:

$$D_\lambda\{3\}\{3\}^{(4)} = \begin{vmatrix} D_4 & D_5 & D_6 & D_7 & D_8 \\ D_3 & D_4 & D_5 & D_6 & D_7 \\ D_2 & D_3 & D_4 & D_5 & D_6 \\ . & D_0 & D_1 & D_2 & D_3 \\ . & . & . & D_0 & D_1 \end{vmatrix} \{3\}\{3\}^{(4)} = \begin{vmatrix} D_1 & D_5 & D_6 & D_7 & D_8 \\ D_0 & D_4 & D_5 & D_6 & D_7 \\ . & D_3 & D_4 & D_5 & D_6 \\ . & D_0 & D_1 & D_2 & D_3 \\ . & . & . & D_0 & D_1 \end{vmatrix} \{3\}^{(4)}.$$

Every D_i with a suffix divisible by 4 is now put equal to unity, and every other D_i is replaced by zero. The required coefficient is thus

$$\begin{vmatrix} . & . & . & . & 1 \\ 1 & 1 & . & . & . \\ . & . & 1 & . & . \\ . & 1 & . & . & . \\ . & . & . & 1 & . \end{vmatrix} = -1.$$

The required criteria in the general case are obtained by considering the various types of determinant which may arise in this way. They are embodied in the following theorem:

Theorem 13. (i) *If the sequence* $\lambda_1-1, \lambda_2-2, \ldots, \lambda_n-n \pmod{n-1}$ *does not contain a complete set of residues* mod $n-1$, *then* $D_\lambda\{m\}\{m\}^{(n-1)} = 0$.

(ii) *If the sequence in* (i) *contains a complete set of residues* (mod $n-1$), *and the two equal residues which necessarily occur correspond to* λ_r-r *and* λ_s-s, *then* $D_\lambda\{m\}\{m\}^{(n-1)} = 0$ *if* $\lambda_r-r+1-m$ *and* $\lambda_s-s+1-m$ *have the same sign, zero being regarded as positive.*

(iii) *If* $\lambda_r-r+1-m \equiv 0 \pmod{n-1}$ *and is non-negative, and* $\lambda_s-s+1-m \equiv 0 \pmod{n-1}$ *and is negative, then* $D_\lambda\{m\}\{m\}^{(n-1)} = (-1)^{s+\theta}$, *where* θ *is* 0 *or* 1 *according as the sequence in* (i), *deprived of* λ_s-s, *is a positive or negative permutation of* $0, n-2, n-3, \ldots, 2, 1$.

Proof. (i) In this case, if p is missing from the complete residue system, then none of λ_r-r-p, for $r = 1, 2, \ldots, n$, will be divisible by $n-1$, and so the final determinant will have a column of zeros in the qth place, where $2 \leqslant q \leqslant n$ and $q \equiv -p \pmod{n-1}$.

(ii) If $\lambda_r-r \equiv \lambda_s-s \equiv x \pmod{n-1}$, then since the sequence in (i) has a complete set of residues mod $n-1$, it follows that

$$\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_n \equiv 1+2+\ldots+n+0+1+2+\ldots+n-2+x \pmod{n-1}.$$

Hence $nm \equiv x+n \pmod{n-1}$, and

$$\lambda_r-r+1-m \equiv \lambda_s-s+1-m \equiv x+1-m \equiv 0 \pmod{n-1}.$$

If $\lambda_r-r+1-m$ and $\lambda_s-s+1-m$ are both negative, the final determinant will have its first column consisting entirely of zeros. If they are both positive, the two units which appear in the rth and sth rows of the first column can be eliminated by subtraction of columns. In either case the value of the determinant is zero.

(iii) By subtracting appropriate columns from the first column, the latter reduces to a column of zeros except for -1 in the sth row. The minor of this element will be $+1$ or -1 according as $\lambda_1-1, \lambda_2-2, \ldots, \lambda_{s-1}-s+1, \lambda_{s+1}-s-1, \ldots, \lambda_n-n \pmod{n-1}$, is a positive or a negative permutation of $0, n-2, n-3, \ldots, 2, 1$. The result follows.

Corollary 1. (i) *When* $\lambda_2 \leqslant m$, *then* $D_\lambda\{m\}\{m\}^{(n-1)} = 0$ *if* $\lambda_1 \not\equiv m \pmod{n-1}$, *or if* $\lambda_2-2, \lambda_3-3, \ldots, \lambda_n-n$ *is not a complete set of residues* (mod $n-1$).

(ii) *When* $\lambda_2 \leqslant m$, *and* $\lambda_1 \equiv m \pmod{n-1}$, *then* $D_\lambda\{m\}\{m\}^{(n-1)}$ *is* $+1$ *or* -1 *according as* $\lambda_2-2, \lambda_3-3, \ldots, \lambda_n-n \pmod{n-1}$ *is a positive or negative permutation of* $0, n-2, n-3, \ldots, 2, 1$.

Corollary 2. *If* $\lambda_2 \leqslant m$, *then*

$$D_{\lambda_1+n, \lambda_2, \lambda_3, \ldots, \lambda_n}\{m+1\}\{m+1\}^{(n-1)} = D_\lambda\{m\}\{m\}^{(n-1)}.$$

THEOREM 14. $D_{2m-\bar{\lambda}}\{m\}\{m\}^{(n-1)} = D_\lambda\{m\}\{m\}^{(n-1)}$.

Proof. (i) $2m-\lambda_{n-t+1}-t = 2m-2-[\lambda_{n-t+1}-(n-t+1)]-(n-1)$, and so if the sequence λ_t-t, mod $n-1$, does not contain a complete residue system, then neither does the sequence $2m-\lambda_{n-t+1}-t$.

(ii) Also, if one of the sequences does contain a complete residue system mod $n-1$ so does the other. Further, if λ_t^* denotes $2m-\lambda_{n-t+1}$, then

$$\lambda^*_{n-r+1}-(n-r+1)+(1-m) = 2m-\lambda_r-n+r-m = (1-n)-[\lambda_r-r+(1-m)].$$

If $\lambda_r-r+(1-m) = k(n-1)$, then

$$\lambda^*_{n-r+1}-(n-r+1)+(1-m) = -(n-1)(k+1).$$

Hence $\lambda^*_{n-r+1}-(n-r+1)+(1-m)$ is a negative or a non-negative multiple of $n-1$ according as $\lambda_r-r+(1-m)$ is a non-negative or negative multiple of $n-1$. Hence if the first column of $D_\lambda\{m\}\{m\}^{(n-1)}$ can be reduced to zeros, so can the first column of $D_{2m-\bar{\lambda}}\{m\}\{m\}^{(n-1)}$.

(iii) If $\lambda_r-r+(1-m)\equiv\lambda_s-s+(1-m)\equiv 0 \pmod{n-1}$, the former being non-negative and the latter negative, then by (ii) above,

$$\lambda^*_{n-r+1}-(n-r+1)+(1-m)\equiv\lambda^*_{n-s+1}-(n-s+1)+(1-m)\equiv 0 \quad (\bmod\, n-1),$$

the first expression being negative and the second non-negative. The sequence λ_t^*-t, $t = 1, 2, \ldots, n-r, n-r+2, \ldots, n$ is the same as the sequence

$$2m-2-[\lambda_{n-t+1}-(n-t+1)] \quad (\bmod\, n-1),$$

and its parity has to be compared with that of

$$\lambda_1-1, \lambda_2-2, \ldots, \lambda_{s-1}-s+1, \lambda_{s+1}-s-1, \ldots, \lambda_n-n \quad (\bmod\, n-1).$$

A cyclic permutation of order $s-r$ is introduced when λ_s-s is reintroduced to the latter sequence and λ_r-r is withdrawn. This gives a sign change of $(-1)^{s-r-1}$. The addition of $2m-2$ has no effect on the relative parity. Replacing λ_t-t by $\lambda_{n-t+1}-(n-t+1)$ gives $\frac{1}{2}(n-1)$ transpositions when n is odd, and $\frac{1}{2}(n-2)$ when n is even. Reversing the sign of $\lambda_{n-t+1}-(n-t+1)$ gives $\frac{1}{2}(n-3)$ transpositions when n is odd, and $\frac{1}{2}(n-2)$ transpositions if n is even. The two operations, of replacement and reversal of sign, together give a sign change of $(-1)^n$ whether n is odd or even.

Finally, expanding the determinant from the $(n-r+1)$th unit in the first column instead of from the sth gives a change of sign of $(-1)^{n-r+1-s}$. The total change of sign involved in passing from one residue system to the other is thus $(-1)^{s-r-1+n+n-r+1-s} = +1$. Hence $D_{2m-\bar{\lambda}}\{m\}\{m\}^{(n-1)} = D_\lambda\{m\}\{m\}^{(n-1)}$.

THEOREM 15. $D_{\lambda_1+1,\,\lambda_2+1,\ldots,\,\lambda_n+1}\{m+1\}\{m+1\}^{(n-1)} = (-1)^n D_\lambda\{m\}\{m\}^{(n-1)}$.

Proof. The determinants $D_{\lambda_1+1,\,\lambda_2+1,\ldots,\,\lambda_n+1}\{m+1\}$ and $D_\lambda\{m\}$ have the same first column, and the addition of a unit to every suffix in the remaining columns of $D_\lambda\{m\}$ is equivalent, in the final stage of the evaluation of the determinant, to a cyclic permutation of these $n-1$ columns. This gives rise to the term $(-1)^n$, and so the result follows by theorem 13.

THEOREM 16. *If* $\lambda_2\leqslant m$ *and* $m-\lambda_2\geqslant\alpha\geqslant-\lambda_n$, *then* $D_{\lambda_\alpha}\{m\}\{m\}^{(n-1)} = (-1)^{n\alpha} D_\lambda\{m\}\{m\}^{(n-1)}$.

Proof. The addition of a unit to each of λ_r-r, for $r = 2, 3, \ldots, n$, is ultimately equivalent to a cyclic permutation of the last $n-1$ columns and so gives a change of sign of $(-1)^n$. The addition of α gives the sign $(-1)^{n\alpha}$. This, with theorem 13, corollary 1, proves the theorem.

COROLLARY 1. *If* $\lambda_2 \leqslant m$, $m-\lambda_2 \geqslant \alpha \geqslant -\lambda_n$ *and* $\lambda_1-(n-1)\alpha \leqslant 2m$, *then*

$$D_{2m-\bar{\lambda}_\alpha}\{m\}\{m\}^{(n-1)} = (-1)^{n\alpha} D_\lambda\{m\}\{m\}^{(n-1)}.$$

THEOREM 17. *If* $\lambda_2 \leqslant m$ *and* $m+\lambda_n \geqslant \beta \geqslant \lambda_2$, *then* $D_{\lambda'_\beta}\{m\}\{m\}^{(n-1)} = (-1)^{n\beta} D_\lambda\{m\}\{m\}^{(n-1)}$.

Proof.

$$\begin{aligned}\lambda'_1 &= 2nm-(n-1)\beta-\lambda_1\\ &\equiv (2n-2)m+2m-\lambda_1 \pmod{n-1}\\ &\equiv 2m-\lambda_1 \pmod{n-1}.\end{aligned}$$

Hence λ_1, λ'_1 are both congruent to m, or both non-congruent to $m \pmod{n-1}$. If

$$\lambda_r - r \; (r = 2, 3, \ldots, n)$$

is not a complete set of residues $\pmod{n-1}$, neither is $\lambda'_r - r$, for $r = 2, 3, \ldots, n$, since

$$\lambda'_r - r = \beta - \lambda_{n-r+2} - r = \beta - [\lambda_{n-r+2}-(n-r+2)] - n - 2.$$

If $\lambda_r - r$ does give a complete set of residues, then so does $\lambda'_r - r$. The change of the set $\lambda_r - r$ into the set $\lambda_{n-r+2}-(n-r+2)$ arises from $\frac{1}{2}(n-1)$ transpositions when n is odd, and from $\frac{1}{2}(n-2)$ transpositions when n is even. The reversal of sign of $\lambda_{n-r+2}-(n-r+2)$ is effected by $\frac{1}{2}(n-3)$ transpositions when n is odd, and by $\frac{1}{2}(n-2)$ transpositions when n is even. The resultant change of sign due to the two operations is thus $(-1)^n$, for both even and odd n. The addition of $\beta-3$ gives a cyclic permutation of order $n-1$ performed $\beta-3$ times on the residues, and so gives a sign change of $(-1)^{n(\beta-1)}$. Hence the total change of parity between $\lambda_r - r$ and $\lambda'_r - r$ is $(-1)^{n\beta}$. This, with theorem 13, corollary 1, proves the theorem.

COROLLARY 1. *If* $\lambda_2 \leqslant m$, $m+\lambda_n \geqslant \beta \geqslant \lambda_2$, *and* $(n-1)(2m-\beta) \leqslant \lambda_1$, *then*

$$D_{2m-\bar{\lambda}'_\beta}\{m\}\{m\}^{(n-1)} = (-1)^{n\beta} D_\lambda\{m\}\{m\}^{(n-1)}.$$

6. EVALUATION OF $D_\lambda\{m\}^{(2)}\{m\}^{(2)}$

THEOREM 18.

$$\begin{aligned}D_\lambda\{m\}^{(2)}\{m\}^{(2)} &= \min[\tfrac{1}{2}(\lambda_1-\lambda_3+2), \tfrac{1}{2}(\lambda_2-\lambda_4+2)] \text{ when } \lambda_1,\lambda_2,\lambda_3,\lambda_4 \text{ are } o,o,o,o \text{ or } e,e,e,e,\\ &= -\tfrac{1}{2}(\lambda_2-\lambda_3+1) \text{ when } \lambda_1,\lambda_2,\lambda_3,\lambda_4 \text{ are } o,o,e,e, \text{ or } e,e,o,o,\\ &= -\min[\tfrac{1}{2}(\lambda_1-\lambda_2+1), \tfrac{1}{2}(\lambda_3-\lambda_4+1)] \text{ when } \lambda_1,\lambda_2,\lambda_3,\lambda_4 \text{ are } o,e,e,o, \text{ or } e,o,o,e,\\ &= 0, \text{ otherwise.}\end{aligned}$$

Proof. Suppose first that all of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are even. Then

$$D_\lambda\{m\}^{(2)}\{m\}^{(2)} = \begin{vmatrix} D^{(2)}_{\frac{1}{2}\lambda_1} & \cdot & D^{(2)}_{\frac{1}{2}(\lambda_1+2)} & \cdot \\ \cdot & D^{(2)}_{\frac{1}{2}\lambda_2} & \cdot & D^{(2)}_{\frac{1}{2}(\lambda_2+2)} \\ D^{(2)}_{\frac{1}{2}(\lambda_3-2)} & \cdot & D^{(2)}_{\frac{1}{2}\lambda_3} & \cdot \\ \cdot & D^{(2)}_{\frac{1}{2}(\lambda_4-2)} & \cdot & D^{(2)}_{\frac{1}{2}\lambda_4}\end{vmatrix} \{m\}^{(2)}\{m\}^{(2)}$$

$$= D^{(2)}_{\frac{1}{2}\lambda_1, \frac{1}{2}\lambda_3} D^{(2)}_{\frac{1}{2}\lambda_2, \frac{1}{2}\lambda_4}\{m\}^{(2)}\{m\}^{(2)}.$$

The product $\{m\}\{m\}$ will give those S-functions $\{2m\}, \{2m-1, 1\}, \{2m-2, 2\}, \ldots, \{m, m\}$ corresponding to all one-rowed and two-rowed partitions of $2m$. The only significant operators occurring in $D^{(2)}_{\frac{1}{2}\lambda_1, \frac{1}{2}\lambda_3} D^{(2)}_{\frac{1}{2}\lambda_2, \frac{1}{2}\lambda_4}$ are thus those which belong to S-functions having one or two rows. Consider the product $\{p, q\}\{r, s\}$, where p, q, r, s are non-negative and

$p \geq q$ and $r \geq s$. If $r-s \leq p-q$, then the Littlewood-Richardson rule shows that the number of S-functions corresponding to one- and two-rowed partitions in the product is $r-s+1$, and is $p-q+1$ when $r-s>p-q$. Hence when $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are even,

$$D_\lambda\{m\}^{(2)}\{m\}^{(2)} = \min\left[\tfrac{1}{2}(\lambda_1-\lambda_3+2), \tfrac{1}{2}(\lambda_2-\lambda_4+2)\right].$$

Similarly, when $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are odd,

$$\begin{aligned} D_\lambda\{m\}^{(2)}\{m\}^{(2)} &= D^{(2)}_{\frac{1}{2}(\lambda_1+1),\,\frac{1}{2}(\lambda_3+1)} D^{(2)}_{\frac{1}{2}(\lambda_2-1),\,\frac{1}{2}(\lambda_4-1)}\{m\}^{(2)}\{m\}^{(2)} \\ &= \min\left[\tfrac{1}{2}(\lambda_1-\lambda_3+2), \tfrac{1}{2}(\lambda_2-\lambda_4+2)\right]. \end{aligned}$$

When λ_1, λ_2 are odd, and λ_3, λ_4 are even,

$$\begin{aligned} D_\lambda\{m\}^{(2)}\{m\}^{(2)} &= \begin{vmatrix} \cdot & D^{(2)}_{\frac{1}{2}(\lambda_1+1)} & \cdot & D^{(2)}_{\frac{1}{2}(\lambda_1+3)} \\ D^{(2)}_{\frac{1}{2}(\lambda_2-1)} & \cdot & D^{(2)}_{\frac{1}{2}(\lambda_2+1)} & \cdot \\ D^{(2)}_{\frac{1}{2}(\lambda_3-2)} & \cdot & D^{(2)}_{\frac{1}{2}\lambda_3} & \cdot \\ \cdot & D^{(2)}_{\frac{1}{2}(\lambda_4-2)} & \cdot & D^{(2)}_{\frac{1}{2}\lambda_4} \end{vmatrix} \{m\}^{(2)}\{m\}^{(2)} \\ &= -D^{(2)}_{\frac{1}{2}(\lambda_1+1),\,\frac{1}{2}\lambda_4} D^{(2)}_{\frac{1}{2}(\lambda_2-1),\,\frac{1}{2}\lambda_3}\{m\}^{(2)}\{m\}^{(2)} \\ &= -\min\left[\tfrac{1}{2}(\lambda_1-\lambda_4+3), \tfrac{1}{2}(\lambda_2-\lambda_3+1)\right] \\ &= -\tfrac{1}{2}(\lambda_2-\lambda_3+1). \end{aligned}$$

Consideration of the various cases that arise leads in a similar way to the remaining results stated in the theorem.

Corollary 1. $D_{\lambda_1+1,\,\lambda_2+1,\,\lambda_3+1,\,\lambda_4+1}\{m+1\}^{(2)}\{m+1\}^{(2)} = D_\lambda\{m\}^{(2)}\{m\}^{(2)}$.

Corollary 2. *If* $\lambda_1 \leq 2m$, *then* $D_{2m-\lambda_4,\,2m-\lambda_3,\,2m-\lambda_2,\,2m-\lambda_1}\{m\}^{(2)}\{m\}^{(2)} = D_\lambda\{m\}^{(2)}\{m\}^{(2)}$.

Corollary 3. *If* $\lambda_2 \leq m$,

$$\begin{aligned} D_\lambda\{m\}^{(2)}\{m\}^{(2)} &= \tfrac{1}{2}(\lambda_2-\lambda_4+2) \text{ when } \lambda_2, \lambda_3, \lambda_4 \text{ are } o, o, o \text{ or } e, e, e, \\ &= -\tfrac{1}{2}(\lambda_2-\lambda_3+1) \text{ when } \lambda_2, \lambda_3, \lambda_4 \text{ are } o, e, e \text{ or } e, o, o, \\ &= -\tfrac{1}{2}(\lambda_3-\lambda_4+1) \text{ when } \lambda_2, \lambda_3, \lambda_4 \text{ are } e, e, o \text{ or } o, o, e, \\ &= 0, \text{ otherwise.} \end{aligned}$$

Corollary 4. *If* $\lambda_2 \leq m$, *then* $D_{\lambda_1+4,\,\lambda_2,\,\lambda_3,\,\lambda_4}\{m+1\}^{(2)}\{m+1\}^{(2)} = D_\lambda\{m\}^{(2)}\{m\}^{(2)}$.

Corollary 5. *With the conditions of the corollaries of theorem* 9 *on* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \alpha, \beta$, *then*

$$D_{\lambda_\alpha}\{m\}^{(2)}\{m\}^{(2)} = D_{\lambda'_\beta}\{m\}^{(2)}\{m\}^{(2)} = D_{2m-\bar{\lambda}_\alpha}\{m\}^{(2)}\{m\}^{(2)} = D_{2m-\bar{\lambda}'_\beta}\{m\}^{(2)}\{m\}^{(2)} = D_\lambda\{m\}^{(2)}\{m\}^{(2)}.$$

7. Evaluation of $D_\lambda\{m\}^{(n)}$

Methods of determining this coefficient have been given by Todd (1949), Duncan (1952 *b*), Littlewood (1951) and Foulkes (1951). Use of the operator technique gives the following simple means of finding the coefficient of $\{\lambda\}$ in $\{m\} \otimes S_n$. This result and the first corollary have previously been obtained by Todd (1949).

Theorem 19. $D_\lambda\{m\}^{(n)} = 0$, *if* $\lambda_r - r \equiv \lambda_s - s \pmod n$, *for some* r, s,

$= \pm 1$, *according as* $\lambda_1, \lambda_2-1, \lambda_3-2, \ldots, \lambda_n-n+1 \pmod n$ *form an even or an odd permutation of* $0, n-1, n-2, \ldots, 2, 1$.

Proof. In the determinant form of D_λ, one and only one suffix in each row will be divisible by n. $D_\lambda\{m\}^{(n)}$ will be zero if and only if two suffixes in the same column are divisible by n, that is, if and only if r, s exist such that $\lambda_r - r \equiv \lambda_s - s \pmod{n}$.

Suppose $\lambda_1, \lambda_2-1, \lambda_3-2, \ldots, \lambda_n-n+1 \pmod{n}$ are $0, n-1, n-2, \ldots, 1$ in this order. Then

$$D_\lambda\{m\}^{(n)} = \text{diag}\,(D^{(n)}_{\lambda_1/n}, D^{(n)}_{\lambda_2/n}, \ldots, D^{(n)}_{\lambda_n/n})\,\{m\}^{(n)} = 1.$$

If $\lambda_1, \lambda_2-1, \lambda_3-2, \ldots, \lambda_n-n+1 \pmod{n}$ are, say, $0, n-2, n-1, n-3, \ldots, 1$ in this order, then

$$D_\lambda\{m\}^{(n)} = \begin{vmatrix} D^{(n)}_{\lambda_1/n} & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & D^{(n)}_{(\lambda_2+1)/n} & \cdot & \cdots & \cdot \\ \cdot & D^{(n)}_{(\lambda_3-1)/n} & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & D^{(n)}_{\lambda_4/n} & \cdots & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & D^{(n)}_{\lambda_n/n} \end{vmatrix} \{m\}^{(n)}$$

$$= -1.$$

The same argument applies if $\lambda_1, \lambda_2-1, \ldots, \lambda_n-n+1 \pmod{n}$ is any permutation of the residues $0, n-1, n-2, \ldots, 1$, the parity of this permutation being the same as that of the permutation determinant obtained by replacing every non-zero element of the operator determinant by a unit. The theorem follows.

Corollary 1. $D_{\lambda_1+1, \lambda_2+1, \ldots, \lambda_n+1}\{m+1\}^{(n)} = (-1)^{n-1} D_\lambda\{m\}^{(n)}$.

Proof. The addition of a unit to each λ_i gives a cyclic permutation of order n to the residues and so gives the factor $(-1)^{n-1}$.

Corollary 2. $D_{\lambda_1+n, \lambda_2, \lambda_3, \ldots, \lambda_n}\{m+1\}^{(n)} = D_\lambda\{m\}^{(n)}$.

Note that the condition $\lambda_2 \leqslant m$, required in previous results relating to $\{\lambda_1+n, \lambda_2, \lambda_3, \ldots, \lambda_n\}$, is not required here.

Corollary 3. *If a partition* (μ) *of non-increasing parts can be obtained from* (λ) *by adding n to one part and subtracting n from another, then* $D_\mu\{m\}^{(n)} = D_\lambda\{m\}^{(n)}$.

Corollary 4. *If* $\lambda_i + n \leqslant \lambda_{i-1}$, *then* $D_{\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i+n, \lambda_{i+1}, \ldots, \lambda_n}\{m+1\}^{(n)} = D_\lambda\{m\}^{(n)}$.

Corollary 5. $D_{2m-\bar{\lambda}}\{m\}^{(n)} = D_\lambda\{m\}^{(n)}$.

Proof. Writing $\{\lambda\}^* = \{2m-\lambda_n, \ldots, 2m-\lambda_1\}$, then

$$\lambda^*_r - r = 2m - \lambda_{n-r+1} - r = 2m - n - [\lambda_{n-r+1} - (n-r+1)] - 1.$$

Changing $\lambda_r - r$ into $\lambda_{n-r+1} - (n-r+1)$ involves $\frac{1}{2}(n-1)$ transpositions when n is odd and $\frac{1}{2}n$ when n is even. Reversal of the sign of $\lambda_{n-r+1}-(n-r+1)$ involves $\frac{1}{2}(n-1)$ transpositions when n is odd, and $\frac{1}{2}(n-2)$ when n is even. The subtraction of a unit introduces a cyclic permutation of order n into the residues and gives a sign of $(-1)^{n-1}$. The result of all the changes of sign is positive in each case.

Corollary 6. *If* $\dfrac{1}{n}(\lambda_1-\lambda_2) \geqslant \alpha \geqslant -\lambda_n$ *and* $\{\lambda\}_\alpha = \{\lambda_1-(n-1)\,\alpha, \lambda_2+\alpha, \ldots, \lambda_n+\alpha\}$, *then* $D_{\lambda_\alpha}\{m\}^{(n)} = (-1)^{(n-1)\,\alpha} D_\lambda\{m\}^{(n)}$.

Note that the conditions $\lambda_2 \leqslant m$, $\lambda_2 \leqslant m-\alpha$ are not required; the only limitation on α is that $\{\lambda\}_\alpha$ should have all its parts positive and in descending order.

Corollary 7. *If* $2m-\frac{1}{n}(\lambda_1-\lambda_n)\geqslant\beta\geqslant\lambda_2$, *and*

$$\{\lambda\}'_\beta=\{2nm-(n-1)\beta-\lambda_1, \beta-\lambda_n, \beta-\lambda_{n-1}, \ldots, \beta-\lambda_2\},$$

then $D_{\lambda'_\beta}\{m\}^{(n)}=(-1)^{(n-1)\beta}D_\lambda\{m\}^{(n)}$.

Proof. The residues mod n are now those of $\beta-\lambda_1, \beta-\lambda_n-1, \beta-\lambda_{n-1}-2, \ldots, \beta-\lambda_2-n+1$. To obtain $\lambda_1, \lambda_n, \lambda_{n-1}, \ldots, \lambda_2$ from $\lambda_1, \lambda_2, \ldots, \lambda_n$ requires $\frac{1}{2}(n-1)$ transpositions when n is odd, and $\frac{1}{2}(n-2)$ when n is even. Reversal of signs throughout requires $\frac{1}{2}(n-1)$ transpositions when n is odd and $\frac{1}{2}(n-2)$ when n is even. The resultant change of sign is thus nil in either case. Addition of β gives the additional sign $(-1)^{(n-1)\beta}$.

The limitations on β are merely to ensure that $\lambda'_1\geqslant\lambda'_2$ and $\lambda'_n\geqslant 0$; they are less stringent than those associated with $\{\lambda\}'_\beta$ in previous results in this paper.

Corollary 8. *If* $\frac{1}{n}(\lambda_1-\lambda_2)\geqslant\alpha\geqslant-\lambda_n$, *and* $\lambda_1-(n-1)\alpha\leqslant 2m$, *and*

$$\{2m-\bar{\lambda}_\alpha\}=\{2m-\lambda_n-\alpha, 2m-\lambda_{n-1}-\alpha, \ldots, 2m-\lambda_1+(n-1)\alpha\},$$

then $D_{2m-\bar{\lambda}_\alpha}\{m\}^{(n)}=(-1)^{(n-1)\alpha}D_\lambda\{m\}^{(n)}$.

Corollary 9. *If* $2m-\frac{1}{n}(\lambda_1-\lambda_n)\geqslant\beta\geqslant\lambda_2$, *and* $\lambda_1\geqslant(n-1)(2m-\beta)$, *and*

$$\{2m-\bar{\lambda}'_\beta\}=\{2m-\beta+\lambda_2, 2m-\beta+\lambda_3, \ldots, 2m-\beta+\lambda_n, \lambda_1-(2m-\beta)(n-1)\},$$

then $D_{2m-\bar{\lambda}'_\beta}\{m\}^{(n)}=(-1)^{(n-1)\beta}D_\lambda\{m\}^{(n)}$.

Theorem 20. *If n is prime,* $D_\lambda\{m\}^{(n)}\equiv D_\lambda\{m\}^n \pmod n$.

Proof. The order of the class $\rho=(1^a2^b3^c\ldots)$ of the symmetric group of order $n!$ is

$$h_\rho=n!/(1^a2^b3^c\ldots a!\,b!\,c!\ldots).$$

When $\rho=(n)$, $h_\rho=(n-1)!$, and since $(n-1)!\equiv-1 \pmod n$ when n is prime, then $h_\rho\equiv-1 \pmod n$. When ρ is any class other than (1^n) or (n), h_ρ will be divisible by n. The theorem follows.

Corollary. *If n is prime,* $D_\lambda\{m\}^n\equiv 1, 0, -1 \pmod n$.

8. Some related coefficients in the same or the conjugate fourth-order plethysm, and in successive fourth-order plethysms

Theorem 21. *If* $\{\Lambda\}_\theta$ *denotes any of the S-functions* $\{\lambda\}_\theta$, $\{\lambda\}'_\theta$, $\{2m-\bar{\lambda}_\theta\}$, $\{2m-\bar{\lambda}'_\theta\}$ *as defined in the corollaries to theorem* 9, *then*

$$D_{\Lambda_\theta}\{m\}\otimes\{\mu\}=D_\lambda\{m\}\otimes\{\mu\} \quad \text{when } \theta \text{ is even,}$$
$$=D_\lambda\{m\}\otimes\{\tilde{\mu}\} \quad \text{when } \theta \text{ is odd,}$$

where (μ) *is any partition of* 4.

Proof. It has been proved that

$$D_{\Lambda_\theta}\{m\}^4=D_\lambda\{m\}^4, \quad D_{\Lambda_\theta}\{m\}^2\{m\}^{(2)}=(-1)^\theta D_\lambda\{m\}^2\{m\}^{(2)},$$

$$D_{\Lambda_\theta}\{m\}\{m\}^{(3)}=D_\lambda\{m\}\{m\}^{(3)}, \quad D_{\Lambda_\theta}\{m\}^{(2)}\{m\}^{(2)}=D_\lambda\{m\}^{(2)}\{m\}^{(2)}, \quad D_{\Lambda_\theta}\{m\}^{(4)}=(-1)^\theta D_\lambda\{m\}^{(4)}.$$

Collecting these results into the expression for $\{m\}\otimes\{\mu\}$ given in §2, and reference to the table of group characters for $n=4$, suffices to prove the theorem.

The following are some simple special cases:

COROLLARY 1. *If* $\lambda_1 \leqslant 2m$,

$$D_{2m-\lambda_4,\,2m-\lambda_3,\,2m-\lambda_2,\,2m-\lambda_1}\{m\}\otimes\{\mu\} = D_\lambda\{m\}\otimes\{\mu\}.$$

COROLLARY 2. *If* $\lambda_2 \leqslant m$,

$$\begin{aligned} D_{m+\lambda_2+\lambda_3+\lambda_4,\,m-\lambda_4,\,m-\lambda_3,\,m-\lambda_2}\{m\}\otimes\{\mu\} &= D_\lambda\{m\}\otimes\{\mu\} \quad \textit{for even } m,\\ &= D_\lambda\{m\}\otimes\{\tilde{\mu}\} \quad \textit{for odd } m.\end{aligned}$$

COROLLARY 3. *If* $\lambda_2+\lambda_3+\lambda_4 \leqslant m$,

$$\begin{aligned} D_{m+\lambda_2,\,m+\lambda_3,\,m+\lambda_4,\,m-(\lambda_2+\lambda_3+\lambda_4)}\{m\}\otimes\{\mu\} &= D_\lambda\{m\}\otimes\{\mu\} \quad \textit{for even } m,\\ &= D_\lambda\{m\}\otimes\{\tilde{\mu}\} \quad \textit{for odd } m.\end{aligned}$$

Some variation is possible in the formulation of these special cases. Thus corollary 1 may be written as

$$D_{2m-a,\,2m-b,\,c,\,d}\{m\}\otimes\{\mu\} = D_{2m-d,\,2m-c,\,b,\,a}\{m\}\otimes\{\mu\},$$

where the only restrictions on a, b, c, d are that all the parts in the partitions are positive and decreasing. Corollary 2 may be written as

$$D_{3m-(b+c-a),\,m-a,\,b,\,c}\{m\}\otimes\{\mu\} = D_{2m+(b+c-a),\,m-c,\,m-b,\,a}\{m\}\otimes\{\mu\}$$

when m is even, with a change of $\{\mu\}$ into $\{\tilde{\mu}\}$ on one side of the equation when m is odd. Similarly, corollary 3 may be expressed as

$$D_{4m-(a+b+c),\,a,\,b,\,c}\{m\}\otimes\{\mu\} = D_{m+a,\,m+b,\,m+c,\,m-(a+b+c)}\{m\}\otimes\{\mu\}$$

when m is even, with the same replacement of $\{\mu\}$ by $\{\tilde{\mu}\}$ on one side when m is odd. Written in this way the results show an interesting form of reciprocity associating $4m$, $3m.m$, $2m.2m$ with $m.m.m.m$, $2m.m.m$, $2m.2m$ respectively.

THEOREM 22. $D_{\lambda_1+1,\,\lambda_2+1,\,\lambda_3+1,\,\lambda_4+1}\{m+1\}\otimes\{\mu\} = D_\lambda\{m\}\otimes\{\tilde{\mu}\}$.

Proof. This follows at once from §2 and theorems 1, 7, 15, 18 (corollary 1) and 19 (corollary 1).

THEOREM 23. *If* $\lambda_2 \leqslant m$, *then* $D_{\lambda_1+4,\,\lambda_2,\,\lambda_3,\,\lambda_4}\{m+1\}\otimes\{\mu\} = D_\lambda\{m\}\otimes\{\mu\}$.

Proof. This follows from §2 and theorems 2 (corollary 6), 9 (corollary 5), 13 (corollary 2), 18 (corollary 4), 19 (corollary 2).

The last three theorems are generalized to any value of n in §10. They are of considerable value in computative work.

9. SPECIFIC FORMULAE FOR SOME OF THE COEFFICIENTS IN $\{m\}\otimes\{\mu\}$, WHERE (μ) IS ANY PARTITION OF 4

THEOREM 24. *If* $m \geqslant k \geqslant 0$, *then* $D_{4m-k,\,k}\{m\}\otimes\{\mu\}$ *is given by tables* 3 *and* 4, *in which the congruences satisfied by k are taken* (mod 12).

TABLE 3

k even

$\{\mu\}$		$k\equiv 0$	$k\equiv 2$ or 10	$k\equiv 4$ or 8	$k\equiv 6$
$\{4\}$	$\frac{1}{48}$	$k^2+12k+48$	$k^2+12k+20$	$k^2+12k+32$	$k^2+12k+36$
$\{31\}$	$\frac{1}{48}$	$3k^2+12k$	$3k^2+12k+12$	$3k^2+12k$	$3k^2+12k+12$
$\{2^2\}$	$\frac{1}{48}$	$2k^2+12k$	$2k^2+12k+16$	$2k^2+12k+16$	$2k^2+12k$
$\{21^2\}$	$\frac{1}{48}$	$3k^2$	$3k^2-12$	$3k^2$	$3k^2-12$
$\{1^4\}$	$\frac{1}{48}$	k^2	k^2-4	k^2-16	k^2+12

TABLE 4

k odd

$\{\mu\}$		$k\equiv 3$	$k\equiv 1$ or 5	$k\equiv 7$ or 11	$k\equiv 9$
$\{4\}$	$\frac{1}{48}$	$k^2+6k+21$	k^2+6k-7	k^2+6k+5	k^2+6k+9
$\{31\}$	$\frac{1}{48}$	$3k^2+18k+15$	$3k^2+18k+27$	$3k^2+18k+15$	$3k^2+18k+27$
$\{2^2\}$	$\frac{1}{48}$	$2k^2-18$	$2k^2-2$	$2k^2-2$	$2k^2-18$
$\{21^2\}$	$\frac{1}{48}$	$3k^2+6k+3$	$3k^2+6k-9$	$3k^2+6k+3$	$3k^2+6k-9$
$\{1^4\}$	$\frac{1}{48}$	k^2-6k+9	k^2-6k+5	k^2-6k-7	$k^2-6k+21$

Proof. From the earlier theorems in this paper it is found without difficulty that

$$D_{4m-k,\,k}\{m\}^4 = \tfrac{1}{2}(k+1)\,(k+2),$$

$$D_{4m-k,\,k}\{m\}^2\,\{m\}^{(2)} = \tfrac{1}{2}(k+2) \quad \text{when } k \text{ is even,}$$

$$= \tfrac{1}{2}(k+1) \quad \text{when } k \text{ is odd,}$$

$$D_{4m-k,\,k}\{m\}\,\{m\}^{(3)} = 0 \quad \text{when } k\equiv 1 \text{ or } 2 \pmod 3,$$

$$= 1 \quad \text{when } k\equiv 0 \pmod 3,$$

$$D_{4m-k,\,k}\{m\}^{(2)}\,\{m\}^{(2)} = \tfrac{1}{2}(k+2) \quad \text{when } k \text{ is even,}$$

$$= -\tfrac{1}{2}(k+1) \quad \text{when } k \text{ is odd,}$$

$$D_{4m-k,\,k}\{m\}^{(4)} = 1 \quad \text{when } k\equiv 0 \pmod 4,$$

$$= -1 \quad \text{when } k\equiv 1 \pmod 4,$$

$$= 0 \quad \text{when } k\equiv 2 \text{ or } 3 \pmod 4.$$

Combining these results for the five partitions (μ) and the twelve possible residues of k gives the tables.

COROLLARY 1. *$D_{4m}\{m\}\otimes\{\mu\} = 1$, when $(\mu) = (4)$ and is zero otherwise.*

COROLLARY 2. (i) *If m is even, $D_{m^4}\{m\}\otimes\{\mu\} = 1$ for $(\mu) = (4)$ and is zero otherwise.*
(ii) *If m is odd, $D_{m^4}\{m\}\otimes\{\mu\} = 1$ for $(\mu) = (1^4)$ and is zero otherwise.*

COROLLARY 3. *Since $\{4m-k, k\}_\alpha = \{4m-k-3\alpha, k+\alpha, \alpha, \alpha\}$, where $m-k\geqslant\alpha\geqslant 0$, then the above tables give $D_{4m-k-3\alpha,\,k+\alpha,\,\alpha,\,\alpha}\{m\}\otimes\{\mu\}$ for all even α in this range. When α is odd, this coefficient is given by $D_{4m-k,\,k}\{m\}\otimes\{\tilde{\mu}\}$ in the tables.*

COROLLARY 4. *Since $\{4m-k, k\}'_\beta = \{4m-3\beta+k, \beta, \beta, \beta-k\}$, where $m\geqslant\beta\geqslant k$, then the tables give $D_{4m-3\beta+k,\,\beta,\,\beta,\,\beta-k}\{m\}\otimes\{\mu\}$ for all even β, and when β is odd, this coefficient is given by*

$$D_{4m-k,\,k}\{m\}\otimes\{\tilde{\mu}\}.$$

The above tables, taken in conjunction with corollaries 3 and 4 and the result that $D_{2m-\lambda_4,\,2m-\lambda_3,\,2m-\lambda_2,\,2m-\lambda_1}\{m\}\otimes\{\mu\} = D_\lambda\{m\}\otimes\{\mu\}$, which is a particular case of theorem 21, are of considerable use in numerical computation. In $\{7\}\otimes\{4\}$, for example, 76 coefficients can be computed from these tables, out of a total of 249. A further selection of coefficients is covered by the following theorem and its corollaries, the results so far obtained accounting for all S-functions in which $\lambda_2\leqslant m$, and all those in which $\lambda_3\geqslant m$.

There is no difficulty in obtaining a specific result for the coefficient of $\{4m-k-r, k, r\}$ in $\{m\}\otimes\{\mu\}$, when (μ) is any partition of 4, but to avoid undue prolixity the following theorem gives results for $\{m\}\otimes\{4\}$ and $\{m\}\otimes\{1^4\}$ only:

THEOREM 25. *If* $m \geqslant k > r > 0$, *then*

$$\text{(i)}\quad D_{4m-k-r,\,k,\,r}\{m\}\otimes\{4\} = \tfrac{1}{24}[\phi+\tfrac{9}{2}(k+2)+\Delta] \quad \textit{when } k+2,\ r+1 \textit{ are } e, o,$$

$$= \tfrac{1}{24}[\phi+\tfrac{3}{2}(\overline{k+2}-\overline{r+1})+\Delta] \quad \textit{when } k+2,\ r+1 \textit{ are } o, o,$$

$$= \tfrac{1}{24}[\phi-\tfrac{9}{2}(r+1)+\Delta] \quad \textit{when } k+2,\ r+1 \textit{ are } o, e,$$

$$= \tfrac{1}{24}[\phi+\Delta] \quad \textit{when } k+2,\ r+1 \textit{ are } e, e;$$

$$\text{(ii)}\quad D_{4m-k-r,\,k,\,r}\{m\}\otimes\{1^4\} = \tfrac{1}{24}[\phi-\tfrac{3}{2}(k+2)+\Gamma] \quad \textit{when } k+2,\ r+1 \textit{ are } e, o,$$

$$= \tfrac{1}{24}[\phi-\tfrac{9}{2}(\overline{k+2}-\overline{r+1})+\Gamma] \quad \textit{when } k+2,\ r+1 \textit{ are } o, o,$$

$$= \tfrac{1}{24}[\phi+\tfrac{3}{2}(r+1)+\Gamma] \quad \textit{when } k+2,\ r+1 \textit{ are } o, e,$$

$$= \tfrac{1}{24}[\phi+\Gamma] \quad \textit{when } k+2,\ r+1 \textit{ are } e, e,$$

where $\phi = \frac{1}{2}(k+2)\,[(k+2)-(r+1)]\,(r+1)$, *and* Δ, Γ *are read from the parts of table* 5 *above and below the principal diagonal respectively. In the table,* $r+1$ *and* $k+2$ *are reduced* (mod 12) *and the reading for* Δ *when* $r+1\equiv p$, $k+2\equiv q$, $p>q$ *is that of* Δ *when* $r+1\equiv q$, $k+2\equiv p$, *with its sign changed. Similarly the reading for* Γ *when* $r+1\equiv p$, $k+2\equiv q$, $p<q$ *is that of* Γ *for* $r+1\equiv q$ *and* $k+2\equiv p$, *with its sign changed.*

TABLE 5

$r+1$ \ $k+2$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	14	−6	0	8	6	−6	8	0	6	2
2	0	−2	0	6	−8	−6	0	−2	0	−6	−8	6
3	0	−6	6	0	0	6	−6	0	0	6	−6	0
4	0	0	8	0	0	8	0	0	8	0	0	8
5	0	−8	−6	6	−8	0	6	−14	0	0	−2	−6
6	0	6	0	−6	0	6	0	6	0	−6	0	6
7	0	−6	14	0	0	2	6	0	8	6	−6	8
8	0	−8	0	0	−8	0	0	−8	0	0	−8	0
9	0	0	−6	6	0	0	−6	6	0	0	6	−6
10	0	6	8	−6	0	14	0	−6	8	6	0	14
11	0	−14	6	0	−8	−6	6	−8	0	−6	−2	0

Proof. From the theorems proved earlier,

$$D_\lambda\{m\}^4 = \tfrac{1}{2}(k-r+1)\,(k+2)\,(r+1) = \phi,$$

$$D_\lambda\{m\}^2\{m\}^{(2)} = \tfrac{1}{2}(k+2) \quad \text{if} \quad k+2, r+1 \text{ are } e, o,$$

$$= \tfrac{1}{2}[(k+2)-(r+1)] \quad \text{if} \quad k+2,\ r+1 \text{ are } o, o,$$

$$= -\tfrac{1}{2}(r+1) \quad \text{if} \quad k+2,\ r+1 \text{ are } o, e,$$

$$= 0 \quad \text{if} \quad k+2,\ r+1 \text{ are } e, e.$$

$D_\lambda\{m\}\{m\}^{(3)}$ is given by table 6.

TABLE 6

$r+1$ \ $k+2$	0	1	2 (mod 3)
0	0	0	0
1	0	0	1
2	0	-1	0

$$D_\lambda\{m\}^{(2)}\{m\}^{(2)} = \tfrac{1}{2}(k+2) \quad \text{when } k+2,\ r+1 \text{ are } e, o,$$
$$= -\tfrac{1}{2}[(k+2)-(r+1)] \quad \text{when } k+2,\ r+1 \text{ are } o, o,$$
$$= -\tfrac{1}{2}(r+1) \quad \text{when } k+2,\ r+1 \text{ are } o, e,$$
$$= 0 \quad \text{when } k+2,\ r+1 \text{ are } e, e.$$

$D_\lambda\{m\}^{(4)}$ is given by table 7.

TABLE 7

$r+1$ \ $k+2$	0	1	2	3 (mod 4)
0	0	0	0	0
1	0	0	1	-1
2	0	-1	0	1
3	0	1	-1	0

Hence

$$D_\lambda\{m\}\otimes\{4\} = \frac{1}{4!}[D_\lambda\{m\}^4 + 6D_\lambda\{m\}^2\{m\}^{(2)} + 8D_\lambda\{m\}\{m\}^{(3)} + 3D_\lambda\{m\}^{(2)}\{m\}^{(2)} + 6D_\lambda\{m\}^{(4)}]$$
$$= \frac{1}{4!}[\phi + \tfrac{9}{2}(k+2) + 8\rho + 6\sigma]$$

for $k+2$, $r+1$ even and odd respectively, where ρ, σ are given by the foregoing tables for $D_\lambda\{m\}\{m\}^{(3)}$ and $D_\lambda\{m\}^{(4)}$ respectively. The table giving $\Delta = 8\rho+6\sigma$ for all k, r is found to be skew-symmetric and is given by the top right-hand triangle of table 5. Similarly, the number $\Gamma = 8\rho - 6\sigma$ occurs in $D_\lambda\{m\}\otimes\{1^4\}$, and the table for Γ is again found to be skew-symmetric, and is given by the bottom left-hand triangle of table 5. Consideration of all possible parities of $k+2$ and $r+1$ suffices to prove the theorem.

COROLLARY. *Since* $\{4m-k-r, k, r\}_\alpha = \{4m-k-r-3\alpha, k+\alpha, r+\alpha, \alpha\}$, *where* $m-k \geq \alpha \geq 0$, *and* $\{4m-k-r, k, r\}'_\beta = \{4m-3\beta+k+r, \beta, \beta-r, \beta-k\}$, *where* $m \geq \beta \geq k$, *then*

$$D_{4m-k-r-3\alpha,\, k+\alpha,\, r+\alpha,\, \alpha}\{m\}\otimes\{4\} \quad \textit{and} \quad D_{4m-3\beta+k+r,\, \beta,\, \beta-r,\, \beta-k}\{m\}\otimes\{4\}$$

are given by (i) *when* α, β *are even, and by* (ii) *when* α, β *are odd. Similarly*

$$D_{4m-k-r-3\alpha,\, k+\alpha,\, r+\alpha,\, \alpha}\{m\}\otimes\{1^4\} \quad \textit{and} \quad D_{4m-3\beta+k+r,\, \beta,\, \beta-r,\, \beta-k}\{m\}\otimes\{1^4\}$$

are given by (ii) *when* α, β *are even, and by* (i) *when* α, β *are odd.*

This theorem and its corollary, together with the equality of the coefficients of $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ and $\{2m-\lambda_4, 2m-\lambda_3, 2m-\lambda_2, 2m-\lambda_1\}$ in $\{m\}\otimes\{\mu\}$, cover a further 67 coefficients in $\{7\}\otimes\{4\}$. Theorems 24 and 25 are perhaps somewhat remarkable in that the results do not involve m explicitly, except in so far as it imposes an upper bound on k. There is no difficulty in obtaining explicit formulae such as that for $D_{3m-k,\, m+k}\{m\}\otimes\{\mu\}$, but the results have to be stated for $m, k \equiv 0, 1, \ldots, 11 \pmod{12}$ and are therefore somewhat lengthy. These results are not included here.

THEOREM 26. *If* $k \leqslant m$, *then* $D_{m+k,\, m+k,\, m-k,\, m-k}\{m\} \otimes \{\mu\}$ *is given by table* 8, *in which the congruences for k are taken* (mod 6).

TABLE 8

m even $\{\mu\}$	m odd $\{\mu\}$		$k\equiv 0$	$k\equiv 1$	$k\equiv 2$	$k\equiv 3$	$k\equiv 4$	$k\equiv 5$
$\{4\}$	$\{1^4\}$	$\frac{1}{6}$	$k+6$	$k-1$	$k+4$	$k+3$	$k+2$	$k+1$
$\{31\}$	$\{21^2\}$		0	0	0	0	0	0
$\{2^2\}$	$\{2^2\}$	$\frac{1}{6}$	$2k$	$2k+4$	$2k+2$	$2k$	$2k+4$	$2k+2$
$\{21^2\}$	$\{31\}$		0	0	0	0	0	0
$\{1^4\}$	$\{4\}$	$\frac{1}{6}$	k	$k-1$	$k-2$	$k+3$	$k-4$	$k+1$

Proof. Earlier theorems of this paper give

$$D_{m+k,\, m+k,\, m-k,\, m-k}\{m\}^4 = k+1,$$

$$\begin{aligned} D_{m+k,\, m+k,\, m-k,\, m-k}\{m\}^2\{m\}^{(2)} &= 1 \quad \text{when } m, k \text{ are } e, e,\\ &= -1 \quad \text{when } m, k \text{ are } o, e,\\ &= 0 \quad \text{otherwise,}\end{aligned}$$

$$\begin{aligned} D_{m+k,\, m+k,\, m-k,\, m-k}\{m\}\{m\}^{(3)} &= 1 \quad \text{when } k\equiv 0 \pmod 3,\\ &= -1 \quad \text{when } k\equiv 1 \pmod 3,\\ &= 0 \quad \text{when } k\equiv 2 \pmod 3,\end{aligned}$$

$$D_{m+k,\, m+k,\, m-k,\, m-k}\{m\}^{(2)}\{m\}^{(2)} = k+1,$$

$$\begin{aligned} D_{m+k,\, m+k,\, m-k,\, m-k}\{m\}^{(4)} &= 0 \quad \text{when } k \text{ is odd,}\\ &= 1 \quad \text{when } k, m \text{ are } e, e,\\ &= -1 \quad \text{when } k, m \text{ are } e, o.\end{aligned}$$

Combining these results with §2 gives table 8.

THEOREM 27. *If* $\{\lambda\} = \{m+2k, m+k, m-k, m-2k\}$, *where* $2k \leqslant m$, *then*

$$\left.\begin{aligned} D_\lambda\{m\}\otimes\{4\} &= \tfrac{1}{48}[\Phi+a] \quad \textit{or} \quad \tfrac{1}{48}[\Phi+e]\\ D_\lambda\{m\}\otimes\{31\} &= \tfrac{1}{16}[\Phi-b] \quad \textit{or} \quad \tfrac{1}{16}[\Phi-d]\\ D_\lambda\{m\}\otimes\{21^2\} &= \tfrac{1}{16}[\Phi-d] \quad \textit{or} \quad \tfrac{1}{16}[\Phi-b]\\ D_\lambda\{m\}\otimes\{1^4\} &= \tfrac{1}{48}[\Phi+e] \quad \textit{or} \quad \tfrac{1}{48}[\Phi+a]\end{aligned}\right\} \textit{according as m is even or odd,}$$

$$D_\lambda\{m\}\otimes\{2^2\} = \tfrac{1}{24}[\Phi+c] \quad \textit{for all } m,$$

where $\Phi = (k+1)(2k^2+3k+2)$ *and* a, b, c, d, e *are given by table* 9.

TABLE 9

$k\equiv x$ (mod 12) x	a	b	c	d	e
0 or 4	$15k+46$	$k+2$	$9k-2$	$5k+2$	$3k-2$
1 or 9	$-9k-5$	$k-3$	$-3k-11$	$-3k+1$	$3k+31$
2	$15k+18$	$k-2$	$9k+6$	$5k+6$	$3k-6$
3 or 7	$-9k+7$	$k+1$	$-3k-11$	$-3k-3$	$3k+19$
5	$-9k-21$	$k-3$	$-3k-3$	$-3k+1$	$3k+15$
6 or 10	$15k+34$	$k-2$	$9k-2$	$5k+6$	$3k+10$
8	$15k+30$	$k+2$	$9k+6$	$5k+2$	$3k-18$
11	$-9k-9$	$k+1$	$-3k-3$	$-3k-3$	$3k+3$

Proof. $D_\lambda\{m\}^4 = \frac{1}{2}(k+1)(2k^2+3k+2) = \frac{1}{2}\Phi$,

$$D_\lambda\{m\}^2\{m\}^{(2)} = \tfrac{1}{2}(k+2) \quad \text{when } m, k \text{ are } e, e,$$
$$= \tfrac{1}{2}(k+1) \quad \text{when } m, k \text{ are } o, o,$$
$$= -\tfrac{1}{2}(k+1) \quad \text{when } m, k \text{ are } e, o,$$
$$= -\tfrac{1}{2}(k+2) \quad \text{when } m, k \text{ are } o, e,$$

$$D_\lambda\{m\}\{m\}^{(3)} = 1 \quad \text{if} \quad k \equiv 0 \text{ or } 1 \pmod 3,$$
$$= 0 \quad \text{if} \quad k \equiv 2 \pmod 3,$$

$$D_\lambda\{m\}^{(2)}\{m\}^{(2)} = \tfrac{1}{2}(3k+2) \quad \text{when } m, k \text{ are } e, e \text{ or } o, e,$$
$$= -\tfrac{1}{2}(k+1) \quad \text{when } m, k \text{ are } o, o \text{ or } e, o.$$

Finally,
$$D_\lambda\{m\}^{(4)} = 1 \quad \text{when } m \text{ is even and } k \equiv 0 \pmod 4,$$
$$= 1 \quad \text{when } m \text{ is odd and } k \equiv 1 \pmod 4,$$
$$= -1 \quad \text{when } m \text{ is even and } k \equiv 1 \pmod 4,$$
$$= -1 \quad \text{when } m \text{ is odd and } k \equiv 0 \pmod 4,$$
$$= 0, \text{ otherwise.}$$

The theorem follows.

Specific formulae may be obtained in the same way for

$$D_\lambda\{m\}\otimes\{\mu\} \quad \text{when} \quad \{\lambda\} = \{m+rk, m+sk, m-sk, m-rk\},$$
$$sk \leqslant m, \quad rk \leqslant m, \quad r > s \quad \text{and} \quad r, s = 0, 1, 2, \ldots,$$

but these results are not given here.

10. General theorems on related coefficients

This section contains generalizations of the theorems of §8.

Theorem 28. $D_{\lambda_1+1, \lambda_2+1, \ldots, \lambda_n+1}\{m+1\}\otimes\{\mu\} = D_{\lambda_1, \lambda_2, \ldots, \lambda_n}\{m\}\otimes\{\tilde{\mu}\}$, *where* (μ) *is any partition of* n.

Proof. Considering only partitions into n or fewer parts,

$$\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\{1^n\} = \{\lambda_1+1, \lambda_2+1, \ldots, \lambda_n+1\}.$$

Hence
$$D_{\lambda_1+1, \lambda_2+1, \ldots, \lambda_n+1}\{m+1\}\otimes\{\mu\} = D_{\lambda_1, \lambda_2, \ldots, \lambda_n} D_{1^n}\{m+1\}\otimes\{\mu\}.$$

But
$$\{m+1\}\otimes\{\mu\} = \frac{1}{n!}\sum_\rho h_\rho \chi_\rho^{(\mu)}\{m+1\}^a[\{m+1\}^{(2)}]^b[\{m+1\}^{(3)}]^c\ldots,$$

where ρ is the class $(1^a 2^b 3^c \ldots)$. The non-zero terms in $D_{1^n}\{m+1\}^a[\{m+1\}^{(2)}]^b[\{m+1\}^{(3)}]^c\ldots$ will arise only from a decomposition of the operator into

$$(D_1 D_1 \ldots \text{to } a \text{ factors})\ (D_{1^2} D_{1^2} \ldots \text{to } b \text{ factors}) \ldots,$$

since $D_{1^p}\{m+1\}^{(r)} = 0$ when $p \neq r$. Hence

$$D_{1^n}\{m+1\}^a[\{m+1\}^{(2)}]^b[\{m+1\}^{(3)}]^c\ldots = [D_1\{m+1\}]^a[D_{1^2}\{m+1\}^{(2)}]^b[D_{1^3}\{m+1\}^{(3)}]^c\ldots$$
$$= \{m\}^a\left[-\frac{\partial}{\partial s_2}\{m+1\}^{(2)}\right]^b\left[\frac{\partial}{\partial s_3}\{m+1\}^{(3)}\right]^c\ldots$$
$$= \pm\{m\}^a[\{m\}^{(2)}]^b[\{m\}^{(3)}]^c\ldots,$$

the sign being $+$ or $-$ according as the class ρ is even or odd. It follows that

$$D_{1^n}\{m+1\}\otimes\{\mu\} = \{m\}\otimes\{\tilde{\mu}\},$$

and so the theorem is proved.

A proof for the case $(\mu)=(n)$ has been given by Newell (1951).

THEOREM 29. *If* $\lambda_2 \leqslant m$, *then*

$$D_{\lambda_1+n,\,\lambda_2,\ldots,\,\lambda_n}\{m+1\}\otimes\{\mu\} = D_{\lambda_1,\,\lambda_2,\ldots,\,\lambda_n}\{m\}\otimes\{\mu\},$$

where (μ) *is any partition of* n.

Proof. It is sufficient to prove that

$$D_{\lambda_1+n,\,\lambda_2,\ldots,\,\lambda_n}\{m+1\}^a\,[\{m+1\}^{(2)}]^b\,[\{m+1\}^{(3)}]^c\ldots = D_{\lambda_1,\,\lambda_2,\ldots,\,\lambda_n}\{m\}^a\,[\{m\}^{(2)}]^b[\{m\}^{(3)}]^c\ldots,$$

where $a+2b+3c+\ldots=n$. If $g_{\eta\nu\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\eta\}\{\nu\}$, where (η), (ν) are respectively partitions of am into not more than a parts, and of $m(2b+3c+\ldots)$ into not more than $2b+3c+\ldots$ parts, then

$$D_{\lambda_1,\,\lambda_2\,\ldots,\,\lambda_n}\{m\}^a\,[\{m\}^{(2)}]^b\,[\{m\}^{(3)}]^c\ldots = \sum_{\eta} g_{\eta\nu\lambda}\,D_{\eta}\{m\}^a\,D_{\nu}([\{m\}^{(2)}]^b\,[\{m\}^{(3)}]^c\ldots).$$

The coefficient of $\{\lambda_1+n, \lambda_2, \ldots, \lambda_n\}$ in $\{\eta_1+a, \eta_2, \ldots, \eta_a\}\{\nu_1+2b+3c+\ldots, \nu_2, \ldots, \nu_{n-a}\}$ will also be $g_{\eta\nu\lambda}$, and so

$$D_{\lambda_1+n,\,\lambda_2,\ldots,\,\lambda_n}\{m+1\}^a\,[\{m+1\}^{(2)}]^b\,[\{m+1\}^{(3)}]^c\ldots$$
$$= \sum_{\eta} g_{\eta\nu\lambda}\,D_{\eta_1+a,\,\eta_2,\ldots,\,\eta_a}\{m+1\}^a\,D_{\nu_1+n-a,\nu_2,\ldots,\,\nu_{n-a}}([\{m+1\}^{(2)}]^b\,[\{m+1\}^{(3)}]^c\ldots).$$

It should be noted that since $\eta_2 \leqslant m$, then every partition of $a(m+1)$ into not more than a parts will be such that the difference between the first two parts will be at least equal to a.

It has been proved in theorem 2 that when $\eta_2 \leqslant m$, $D_{\eta}\{m\}^a = D_{\eta_1+a,\,\eta_2,\ldots,\,\eta_a}\{m+1\}^a$ and so it remains to prove that

$$D_{\nu}([\{m\}^{(2)}]^b\,[\{m\}^{(3)}]^c\ldots) = D_{\nu_1+n-a,\,\nu_2,\ldots,\nu_{n-a}}([\{m+1\}^{(2)}]^b\,[\{m+1\}^{(3)}]^c\ldots).$$

Let $\{\nu\}$ appear in $\{\xi\}\{\zeta\}$ with coefficient $g_{\xi\zeta\nu}$, where (ξ), (ζ) are respectively partitions of $2bm$ into not more than $2b$ parts and of $m(n-a-2b)$ into not more than $n-a-2b$ parts. Then $\{\nu_1+n-a, \nu_2, \ldots, \nu_{n-a}\}$ will appear in $\{\xi_1+2b, \xi_2, \ldots, \xi_{2b}\}\{\zeta_1+n-a-2b, \zeta_2, \ldots\}$ with coefficient $g_{\xi\zeta\nu}$. Hence

$$D_{\nu_1+n-a,\,\nu_2,\ldots,\,\nu_{n-a}}([\{m+1\}^{(2)}]^b\,[\{m+1\}^{(3)}]^c\ldots)$$
$$= \sum_{\xi} g_{\xi\zeta\nu}\,D_{\xi_1+2b,\,\xi_2,\ldots,\,\xi_{2b}}[\{m+1\}^{(2)}]^b\,D_{\zeta_1+n-a-2b,\,\zeta_2,\ldots}([\{m+1\}^{(3)}]^c\ldots).$$

The next step is to show that

$$D_{\xi_1+2b,\,\xi_2,\ldots,\,\xi_{2b}}[\{m+1\}^{(2)}]^b = D_{\xi_1,\,\xi_2,\ldots,\,\xi_{2b}}[\{m\}^{(2)}]^b,$$

and it is evident that to continue the proof in this way it must be shown that

$$D_{\theta_1+pr,\,\theta_2,\ldots,\,\theta_{pr}}[\{m+1\}^{(p)}]^r = D_{\theta_1,\,\theta_2,\ldots,\,\theta_{pr}}[\{m\}^{(p)}]^r,$$

where $(\theta_1, \theta_2, \ldots, \theta_{pr})$ is a partition of mpr. The two operator determinants concerned will differ only in the first row, and when all the suffixes are divided by p and those operator elements with suffixes prime to p are replaced by zeros, the operators reduce to

$$D_{\theta_1, \theta_2, \ldots, \theta_{pr}} = \Sigma \pm D^{(p)}_{\tau_1, \tau_2, \tau_3, \ldots} \quad \text{and} \quad D_{\theta_1+pr, \theta_2, \ldots, \theta_{pr}} = \Sigma \pm D^{(p)}_{\tau_1+r, \tau_2, \ldots},$$

where $(\tau_1, \tau_2, \ldots)$ is a partition of rm. But, as in theorem 2,

$$D^{(p)}_{\tau_1+r, \tau_2, \ldots}[\{m+1\}^{(p)}]^r = D^{(p)}_{\tau_1, \tau_2, \ldots}[\{m\}^{(p)}]^r,$$

and so the theorem is proved.

Theorem 29 is a refinement of a result, proved by Ibrahim (1952) using an invariant theory argument, that

$$D_{\lambda_1+n, \lambda_2, \ldots, \lambda_n}\{m+1\} \otimes \{n\} \geqslant D_{\lambda_1, \lambda_2, \ldots, \lambda_n}\{m\} \otimes \{n\}.$$

Theorems 28 and 29 may now be combined to give a generalization of the part of theorem 21 which refers to $\{\lambda\}_\alpha$.

THEOREM 30. *If* $\lambda_2 \leqslant m$, $m-\lambda_2 \geqslant \alpha \geqslant -\lambda_n$, *and* $\{\lambda\}_\alpha = \{\lambda_1-(n-1)\alpha, \lambda_2+\alpha, \lambda_3+\alpha, \ldots, \lambda_n+\alpha\}$, *then*

$$D_{\lambda_\alpha}\{m\} \otimes \{\mu\} = D_\lambda\{m\} \otimes \{\mu\} \quad \textit{when } \alpha \textit{ is even},$$
$$= D_\lambda\{m\} \otimes \{\tilde{\mu}\} \quad \textit{when } \alpha \textit{ is odd},$$

where (μ) *is any partition of* n.

Proof. From theorem 28,

$$D_{\lambda_1+\alpha, \lambda_2+\alpha, \ldots, \lambda_n+\alpha}\{m+\alpha\} \otimes \{\mu\} = D_{\lambda_1, \lambda_2, \ldots, \lambda_n}\{m\} \otimes \{\mu\} \quad \text{for even } \alpha,$$
$$= D_{\lambda_1, \lambda_2, \ldots, \lambda_n}\{m\} \otimes \{\tilde{\mu}\} \quad \text{for odd } \alpha,$$

and from theorem 29, if $\lambda_2 \leqslant m-\alpha$, then

$$D_{\lambda_1+\alpha-n\alpha, \lambda_2+\alpha, \ldots, \lambda_n+\alpha}\{m\} \otimes \{\mu\} = D_{\lambda_1+\alpha, \lambda_2+\alpha, \ldots, \lambda_n+\alpha}\{m+\alpha\} \otimes \{\mu\},$$

which proves the theorem. Negative values of α are allowable subject to $\lambda_n+\alpha \geqslant 0$. For negative α, theorem 29 requires $\lambda_2+\alpha \leqslant m+\alpha$, giving $\lambda_2 \leqslant m$ in this case.

The next general theorem on the plethysm $\{m\} \otimes \{\mu\}$ requires two preliminary results, theorems 31 and 32.

THEOREM 31. $D_\lambda\{\mu\}\{\nu\} = D_{2m-\bar{\lambda}}\{2m-\bar{\mu}\}\{2m-\bar{\nu}\}$, *where* (λ), (μ), (ν) *are partitions of* mn, ma, mb *into* n, a, b *parts respectively, and* $a+b=n$.

Proof. Consider a rectangle of n rows, each row containing $2m$ nodes. Place the graph of (μ) in the top left-hand corner and the inverted graph of $(2m-\bar{\mu})$ in the bottom right-hand corner. Complete the graph of (λ), assuming this is possible, by adding ν_1 symbols α_1, ν_2 symbols $\alpha_2, \ldots, \nu_b$ symbols α_b in succession to the (μ) graph so that no two of the added symbols with the same suffix are in the same column and the suffixes increase when read down any column. Fill up the remaining nodes in the rectangle, column by column, with $\beta_1, \beta_2, \ldots, \beta_b$, so that the suffixes of the α's and β's in each column are all different, and the suffixes of the β's in any column, reading from the lowest suffix, are in ascending order of magnitude. In this way the inverted graph of $(2m-\bar{\lambda})$ has been constructed in the bottom right-hand corner by adding to the graph of $(2m-\bar{\mu})$, $2m-\nu_1$ symbols β_1, $2m-\nu_2$ symbols β_2, and so on, such that no two of the added symbols with the same suffix are in the same column. To each way of constructing the diagram (λ) in the top left-hand corner there corresponds one way of constructing the diagram $(2m-\bar{\lambda})$ in the opposite corner, and conversely.

It has thus been shown that

$$D_\lambda\{\mu\}\{\nu_1\}\{\nu_2\}\dots\{\nu_b\} = D_{2m-\bar{\lambda}}\{2m-\bar{\mu}\}\{2m-\nu_b\}\{2m-\nu_{b-1}\}\dots\{2m-\nu_1\},$$

and similarly

$$D_\lambda\{\mu\}\{\nu_1+1\}\{\nu_2+1\}\dots\{\nu_{b-1}+1\}\{\nu_b+1-b\}$$
$$= D_{2m-\bar{\lambda}}\{2m-\bar{\mu}\}\{2m-\nu_b-1+b\}\{2m-\nu_{b-1}-1\}\{2m-\nu_{b-2}-1\}\dots\{2m-\nu_1-1\},$$

and so on for every term in the h-determinantal form of $\{\nu\}$. If $\nu_1+k>2m$, then both sides become zero. Expanding the h-determinantal forms of $\{\nu\}$ and $\{2m-\bar{\nu}\}$ as

$$\{\nu\} = \{\nu_1\}\{\nu_2\}\dots\{\nu_b\}-\{\nu_1+1\}\{\nu_2+1\}\dots\{\nu_{b-1}+1\}\{\nu_b+1-b\}+\dots$$

and $$\{2m-\bar{\nu}\} = \{2m-\nu_b\}\{2m-\nu_{b-1}\}\dots\{2m-\nu_1\}$$
$$-\{2m-\nu_b-1+b\}\{2m-\nu_{b-1}-1\}\{2m-\nu_{b-2}-1\}\dots\{2m-\nu_1-1\}+\dots,$$

it follows that $D_\lambda\{\mu\}\{\nu\} = D_{2m-\bar{\lambda}}\{2m-\bar{\mu}\}\{2m-\bar{\nu}\}$.

An example may make the procedure clearer. Take $m=7$, $a=2$, $b=4$, $n=6$,

$$\{\lambda\} = \{11.98752\},\quad \{\mu\} = \{86\},\quad \{\nu\} = \{10.85^2\},$$
$$\{2m-\bar{\lambda}\} = \{12.97653\},\quad \{2m-\bar{\mu}\} = \{86\},\quad \{2m-\bar{\nu}\} = \{9^264\}.$$

One way of constructing the rectangle is then

.	.	.	.	.	.	.	.	α_1	α_1	α_1	β_4	β_4	β_4
.	.	.	.	.	.	α_1	α_2	α_2	β_4	β_4	β_3	β_3	β_3
α_1	α_1	α_1	α_1	α_1	α_1	α_2	α_3	β_4	β_3	β_3	β_2	β_2	β_2
α_2	α_2	α_2	α_2	α_2	α_3	α_3	β_4	β_3	β_2	β_2	β_1	β_1	β_1
α_3	α_3	α_4	α_4	α_4	β_4	β_4	β_1	.	.	.	.	.	.
α_4	α_4	β_3	β_3	β_3	β_2	.	.	.	.	.	.	.	.

the top portion giving one way of forming $\{11.98752\}$ from $\{86\}\{10\}\{8\}\{5\}\{5\}$ and the bottom portion giving the corresponding way of forming $\{12.97653\}$ from $\{86\}\{9\}\{9\}\{6\}\{4\}$. Similarly, a diagram can be drawn associating one way of constructing $\{11.98752\}$ from $\{86\}\{11\}\{9\}\{6\}\{2\}$ with the corresponding method of constructing $\{12.97653\}$ from $\{86\}\{12\}\{8\}\{5\}\{3\}$, and so on.

In the diagram above it so happens that the α's form a lattice permutation

$$\alpha_1^3\alpha_2^2\alpha_1\alpha_3\alpha_2\alpha_1^6\alpha_3^2\alpha_2^5\alpha_4^3\alpha_3^2\alpha_4^2,$$

when read from right to left along the successive rows starting at the first, whereas the β's form a lattice permutation

$$\beta_4^3\beta_3^3\beta_4^2\beta_2^3\beta_3^2\beta_4\beta_1^3\beta_2^2\beta_3\beta_4\beta_1\beta_4^2\beta_2\beta_3^3$$

when read from left to right along the rows starting at the last, both diagrams being supposed in their upright position. It seems probable that an alternative proof of theorem 31 might be based on this observation.

THEOREM 32. *If* (μ) *is a partition of mpr into not more than pr parts, then*

$$D_\mu[\{m\}^{(p)}]^r = D_{2m-\bar{\mu}}[\{m\}^{(p)}]^r.$$

Proof. To convert D_μ into $D_\xi^{(p)}$ the following result, due to Littlewood (1950, p. 133), is used. If the numbers of the sequence

$$[\alpha_i] = [\mu_1+pr-1, \mu_2+pr-2, \mu_3+pr-3, \ldots, \mu_{pr}],$$

congruent respectively to $0, 1, 2, \ldots, p-1 \pmod p$, are not all equal, then $\{\mu\} = 0$. If they are equal and those congruent to $q \pmod p$ are

$$p[\xi_{q1}+r-1]+q, \quad p[\xi_{q2}+r-2]+q, \quad \ldots, \quad p\xi_{qr}+q,$$

then $$\{\mu\} = \theta\{\xi_{01}, \xi_{02}, \ldots, \xi_{0r}\}^{(p)}\{\xi_{11}, \xi_{12}, \ldots, \xi_{1r}\}^{(p)} \ldots \{\xi_{p-1,1}, \ldots, \xi_{p-1,r}\}^{(p)},$$

where θ is $+1$ or -1 according as a certain rearrangement of $[\alpha_i]$ is positive or negative.

The sequences $[\alpha_i]$ written down for $\{\mu\}$ and $\{2m-\bar{\mu}\}$ reduce to

$$\mu_1-1, \quad \mu_2-2, \quad \ldots, \quad \mu_{pr}-pr \quad \pmod p$$

and $$2m-1-(\mu_{pr}-pr), \quad \ldots, \quad 2m-1-(\mu_2-2), \quad 2m-1-(\mu_1-1) \quad \pmod p.$$

These reduced sequences clearly consist of the same numbers, but they will differ in general in their arrangement. It follows that if $\{\mu\}$ is zero, then so is $\{2m-\bar{\mu}\}$, and conversely. The next step is to show that when $\{\mu\}$ is not zero, then θ is the same for both $\{\mu\}$ and $\{2m-\bar{\mu}\}$.

Let $\alpha_i = \mu_i+pr-i \equiv x \pmod p$. Then

$$\beta_{pr-i+1} = (2m-\mu_i)+pr-(pr-i+1) \equiv 2m-1-x \quad \pmod p.$$

If $2m-1 \equiv \epsilon \pmod p$, then $\alpha_i \equiv x \pmod p$ implies $\beta_{pr-i+1} \equiv \epsilon - x \pmod p$. If $\alpha_h, \alpha_k, \ldots, \alpha_s, \alpha_t$ are the first p integers of the sequence $[\alpha_i]$ to be congruent to $p-1, p-2, \ldots, 2, 1, 0 \pmod p$ in order, then $\beta_{pr-t+1}, \beta_{pr-s+1}, \ldots, \beta_{pr-k+1}, \beta_{pr-h+1}$ are the last p integers of the sequence $[\beta_i]$ to be congruent to $\epsilon, \epsilon-1, \epsilon-2, \ldots, \epsilon-(p-1) \pmod p$ in order. If the permutation of $[\alpha_i]$ which rearranges it into r sets, each set congruent to $p-1, p-2, \ldots, 2, 1, 0$ in order, and having the α's congruent to any given residue in descending order throughout the r sets, is an even permutation, then $\theta = +1$. If it is an odd permutation then $\theta = -1$. To such a permutation of $[\alpha_i]$ there corresponds, by changing i into $pr-i+1$, a permutation of the same orders in the cycles of $[\beta_i]$; for example, the cycle or cycles giving the first set of α's congruent to $p-1, p-2, \ldots, 1, 0$ in order correspond to a cycle or cycles of the same order on the β's giving the last set of β's congruent to $\epsilon, \epsilon-1, \ldots, \epsilon-(p-1)$ in order. To show that θ is the same for $[\alpha_i]$ and $[\beta_i]$, all that is now required is to show that $p-1, p-2, \ldots, 2, 1, 0$ can be obtained from $\epsilon, \epsilon-1, \ldots, \epsilon-(p-1) \pmod p$ by an even permutation. This is so because the addition of $(p-1)-\epsilon$ to each term of the latter sequence is equivalent to adding $-(1+\epsilon) \pmod p$, which is $2m \pmod p$. Hence θ is the same for both $\{\mu\}$ and $\{2m-\bar{\mu}\}$.

Let $\{\mu\}$ and $\{2m-\bar{\mu}\}$ be written respectively as

$$\theta\{\xi_{01}, \xi_{02}, \ldots, \xi_{0r}\}^{(p)}\{\xi_{11}, \xi_{12}, \ldots, \xi_{1r}\}^{(p)} \ldots \{\xi_{p-1,1}, \ldots, \xi_{p-1,r}\}^{(p)}$$

and $$\theta\{\bar{\xi}_{01}, \bar{\xi}_{02}, \ldots, \bar{\xi}_{0r}\}^{(p)}\{\bar{\xi}_{11}, \bar{\xi}_{12}, \ldots, \bar{\xi}_{1r}\}^{(p)} \ldots \{\bar{\xi}_{p-1,1}, \ldots, \bar{\xi}_{p-1,r}\}^{(p)}.$$

It will now be shown that

$$\{\xi_{x1}, \xi_{x2}, \ldots, \xi_{xr}\} = \{a_x - \bar{\xi}_{\epsilon-x,r}, a_x - \bar{\xi}_{\epsilon-x,r-1}, \ldots, a_x - \bar{\xi}_{\epsilon-x,1}\},$$

where $a_0+a_1+a_2+\ldots+a_{p-1} = 2m$.

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Let α_{xi} be the ith $\alpha \equiv x \pmod p$, where $x \geqslant 0$, and let $\beta_{\epsilon-x,\, r-i+1}$ be the $(r-i+1)$th $\beta \equiv \epsilon - x \pmod p$, where $2m-1 = \epsilon + pk$ and $\epsilon \geqslant 0$. Then

$$\alpha_{xi} = p[\xi_{xi}+r-i]+x \quad \text{for } x = 0, 1, 2, \ldots, p-1,$$

$$\begin{aligned}\beta_{\epsilon-x,\, r-i+1} &= p[\bar{\xi}_{\epsilon-x,\, r-i+1}+r-(r-i+1)]+\epsilon-x \quad \text{if} \quad x \leqslant \epsilon,\\ &= p[\bar{\xi}_{\epsilon-x,\, r-i+1}+r-(r-i+1)]+p+\epsilon-x \quad \text{if} \quad x > \epsilon,\end{aligned}$$

it being assumed that when $\epsilon - x$ is negative in a suffix of β or ξ it is replaced by $p+(\epsilon-x)$. But if $\alpha_{xi} = \alpha_j$, then $\beta_{\epsilon-x,\, r-i+1} = \beta_{pr-j+1}$, and so $\mu_j + pr - j = p[\xi_{xi}+r-i]+x$, and

$$\begin{aligned}2m-\mu_j+pr-(pr-j+1) &= p[\bar{\xi}_{\epsilon-x,\, r-i+1}+i-1]+\epsilon-x \quad \text{if} \quad x \leqslant \epsilon,\\ &= p[\bar{\xi}_{\epsilon-x,\, r-i+1}+i-1]+p+(\epsilon-x) \quad \text{if} \quad x > \epsilon.\end{aligned}$$

Hence
$$\begin{aligned}p[\xi_{xi}+\bar{\xi}_{\epsilon-x,\, r-i+1}] &= 2m-1+p-\epsilon \quad \text{if} \quad x \leqslant \epsilon,\\ &= 2m-1-\epsilon \quad \text{if} \quad x > \epsilon,\end{aligned}$$

that is,
$$\begin{aligned}\xi_{xi}+\bar{\xi}_{\epsilon-x,\, r-i+1} &= k+1 \quad \text{if} \quad x \leqslant \epsilon,\\ &= k \quad \text{if} \quad x > \epsilon.\end{aligned}$$

If $a_x = k+1$ when $x \leqslant \epsilon$ and $a_x = k$ when $x > \epsilon$, then

$$a_0+a_1+\ldots+a_\epsilon+a_{\epsilon+1}+\ldots+a_{p-1} = (\epsilon+1)(k+1)+(p-\epsilon-1)k = kp+\epsilon+1 = 2m.$$

It has thus been proved that if $\{\mu\} = \theta\{\xi_0\}^{(p)}\{\xi_1\}^{(p)}\ldots\{\xi_{p-1}\}^{(p)}$, then

$$\{2m-\bar{\mu}\} = \theta\{a_0-\bar{\xi}_{\epsilon-0}\}^{(p)}\{a_1-\bar{\xi}_{\epsilon-1}\}^{(p)}\ldots\{a_{p-1}-\bar{\xi}_{\epsilon-p+1}\}^{(p)},$$

where $a_0+a_1+\ldots+a_{p-1} = 2m$.

It must now be shown that if (ξ) is a partition of mr into not more than r parts, then the coefficient of $\{\xi\}$ in $\{\xi_0\}\{\xi_1\}\ldots\{\xi_{p-1}\}$ is the same as the coefficient of $\{2m-\bar{\xi}\}$ in

$$\{a_0-\bar{\xi}_{\epsilon-0}\}\{a_1-\bar{\xi}_{\epsilon-1}\}\ldots\{a_{p-1}-\bar{\xi}_{\epsilon-p+1}\}.$$

The partition graphs of $\{\xi_{\epsilon-x}\}$ and $\{a_x-\bar{\xi}_{\epsilon-x}\}$ are such that they can be placed in a rectangle of r rows of a_x nodes in each row so that $\{\xi_{\epsilon-x}\}$ occupies the top left-hand corner and the inverted graph of $\{a_x-\bar{\xi}_{\epsilon-x}\}$ the bottom right-hand corner, the nodes of the rectangle being completely used up.

Take a rectangle of r rows with $2m$ nodes in each row and place the graph of $\{\xi_{\epsilon-0}\}$ in the top left-hand corner and that of $\{a_0-\bar{\xi}_{\epsilon-0}\}$ inverted in the bottom right-hand corner. The places so filled can be regarded as originating from the rectangle of r rows of a_0 nodes in each row formed from the graph of $\{\xi_{\epsilon-0}\}$ and the inverted graph of $\{a_0-\bar{\xi}_{\epsilon-0}\}$. Consider now the $a_1 \times r$ rectangle made up of $\{\xi_{\epsilon-1}\}$ and $\{a_1-\bar{\xi}_{\epsilon-1}\}$ inverted. Let the nodes of $\{\xi_{\epsilon-1}\}$ be denoted by p_i and those of $\{a_1-\bar{\xi}_{\epsilon-1}\}$ by q_i, where i denotes the column in which the node occurs. Take the symbols p_1 and apply them to the graph of $\{\xi_{\epsilon-0}\}$ in the top left-hand corner of the $2m \times r$ rectangle so that no two of the symbols are in the same row, and so that a Young diagram is formed in the usual way. Then apply the symbols q_1 to the inverted graph of $\{a_0-\bar{\xi}_{\epsilon-0}\}$ in the bottom right-hand corner so that no two symbols q_1 are in the same row, and so that each row of the rectangle contains either a p_1 or a q_1. Continue in the same way with the symbols p_2 and q_2, so that each row contains a p_2 or a q_2, and then with p_3, q_3 and so on.

The placing of the q_i's is thus determined by that of the p_i's, and so in this way the construction of a diagram corresponding to some S-function $\{\pi\}$ of r or fewer parts in the top corner of the rectangle from the product $\{\xi_{\epsilon-0}\}\{1^{c_1}\}\{1^{c_2}\}\ldots\{1^{c_{a_1}}\}$, where c_i is the number of nodes in the ith column of $\{\xi_{\epsilon-1}\}$, has been made to correspond uniquely with the construction, in the bottom corner of the rectangle, of a diagram $\{(a_0+a_1)-\overline{\pi}\}$ from the product $\{a_0-\overline{\xi}_{\epsilon-0}\}\{1^{r-c_1}\}\{1^{r-c_2}\}\ldots\{1^{r-c_{a_1}}\}$. Similarly, to each construction of $\{\pi\}$ from

$$\{\xi_{\epsilon-0}\}\{1^{c_1+1}\}\{1^{c_2+1}\}\ldots\{1^{c_{a_1}-a_1+1}\}$$

there corresponds a unique construction of $\{(a_0+a_1)-\overline{\pi}\}$ from

$$\{a_0-\overline{\xi}_{\epsilon-0}\}\{1^{r-c_1-1}\}\{1^{r-c_2-1}\}\ldots\{1^{r-c_{a_1}+a_1-1}\},$$

and so on for every term in the expansion of the a-determinantal forms of $\{\xi_{\epsilon-1}\}$ and $\{a_1-\overline{\xi}_{\epsilon-1}\}$, that is, the determinants whose elements are of type $\{1^k\}$. The argument now follows that of theorem 31, using the a-determinants instead of the h-determinants used in that theorem. Hence

$$D_{\pi}\{\xi_{\epsilon-0}\}\{\xi_{\epsilon-1}\} = D_{(a_1+a_0)-\overline{\pi}}\{a_0-\overline{\xi}_{\epsilon-0}\}\{a_1-\overline{\xi}_{\epsilon-1}\}.$$

Continuing this process with $\{\xi_{\epsilon-2}\}$ and $\{a_2-\overline{\xi}_{\epsilon-2}\}$ and so on, it follows that since

$$a_0+a_1+\ldots+a_{p-1} = 2m,$$

the nodes of the p smaller rectangles will ultimately exactly fill the $2m\times r$ rectangle, and to each way of constructing $\{\xi\}$ from $\{\xi_0\}\{\xi_1\}\ldots\{\xi_{p-1}\}$ in the top left-hand corner there will correspond one way of constructing $\{2m-\overline{\xi}\}$ from $\{a_0-\overline{\xi}_{\epsilon-0}\}\{a_1-\overline{\xi}_{\epsilon-1}\}\ldots\{a_1-\overline{\xi}_{\epsilon-p+1}\}$ in the bottom right-hand corner and conversely.

It has thus been shown that if

$$D_{\mu} = \theta D^{(p)}_{\xi_0}D^{(p)}_{\xi_1}\ldots D^{(p)}_{\xi_{p-1}} = \sum_{\xi} g_{\mu\xi}D^{(p)}_{\xi},$$

then

$$D_{2m-\overline{\mu}} = \theta D^{(p)}_{a_0-\overline{\xi}_{\epsilon-0}}D^{(p)}_{a_1-\ \epsilon-1}\ldots D^{(p)}_{a_{p-1}-\overline{\xi}_{\epsilon-p+1}} = \sum_{\xi} g_{\mu\xi}D^{(p)}_{2m-\overline{\xi}}.$$

But from theorem 3,

$$D^{(p)}_{\xi}[\{m\}^{(p)}]^r = D^{(p)}_{2m-\overline{\xi}}[\{m\}^{(p)}]^r,$$

since the superscript (p) is immaterial to the way the S-functions and the operators combine together. Hence the proof is complete.

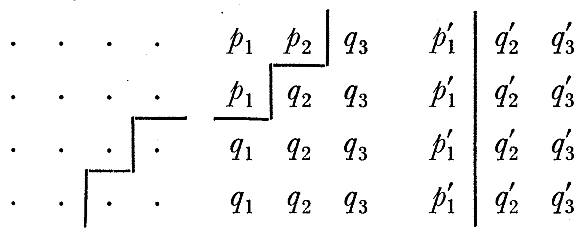
As an illustrative example take $m=5$, $p=3$, $r=4$, $\{\mu\}=\{10.8765^3 43^3 1\}$, and so $\{2m-\overline{\mu}\}=\{97^3 65^3 4320\}$. The sequence $[\alpha_i]$ is 21, 18, 16, 14, 12, 11, 10, 8, 6, 5, 4, 1. This can be arranged as

$$\begin{array}{llll} 3(4+3)+0, & 3(4+2)+0, & 3(3+1)+0, & 3(2+0)+0, \\ 3(2+3)+1, & 3(1+2)+1, & 3(0+1)+1, & 3(0+0)+1, \\ 3(1+3)+2, & 3(1+2)+2, & 3(1+1)+2, & 3(1+0)+2, \end{array}$$

from which $\{10.8765^3 43^3 1\} = \theta\{4^2 32\}^{(3)}\{21\}^{(3)}\{1^4\}^{(3)}$, where θ is $+1$ or -1 according as the permutation coverting $[\alpha_i]$ into 14, 16, 21; 11, 10, 18; 8, 4, 12; 5, 1, 6 is even or odd.

Treating $\{2m-\overline{\mu}\}$ similarly gives $[\beta_i]$ as 20, 17, 16, 15, 13, 11, 10, 9, 7, 5, 3, 0 from which $\{97^3 65^3 4320\} = \theta'\{21\}^{(3)}\{2^4\}^{(3)}\{3^2 21\}^{(3)}$, where θ' is $+1$ or -1 according as the permutation converting $[\beta_i]$ into 20, 16, 15; 17, 13, 9; 11, 10, 3; 5, 7, 0 is even or odd. The permutation on the twelve numbers of $[\alpha_i]$ is (14623) (578.11.12.9) (10) and the corresponding permutation (12.97.11.10) (865214) (3) applied to $[\beta_i]$ gives 15, 20, 16; 9, 17, 13; 3, 11, 10; 0, 5, 7. To put these into the required order clearly requires an even permutation, showing that $\theta = \theta'$.

The three S-functions arising from $\{\mu\}$ can be combined with their counterparts arising from $\{2m-\bar{\mu}\}$ into the rectangles

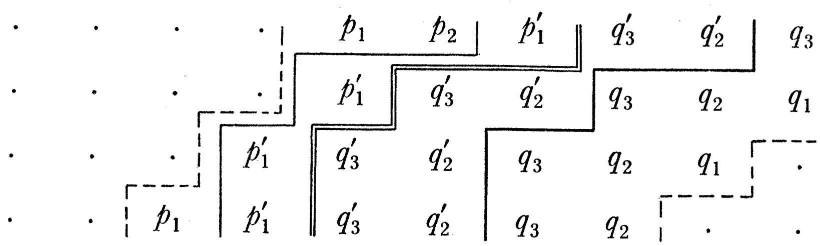


in which $a_0 = 4$, $a_1 = a_2 = 3$, agreeing with $a_x = k+1$ for $x \leqslant \epsilon$, and $a_x = k$ for $x > \epsilon$, since $k = 3$, $\epsilon = 0$. The following diagram shows how one way of forming $\{\pi\} = \{643^2\}$ from $\{\xi_{\epsilon-0}\}\{1^{c_1}\}\{1^{c_2}\}\{1^{c_3}\} = \{4^2 32\}\{1^2\}\{1\}\{1^0\}$ corresponds to one way of forming

$$\{(a_0+a_1)-\bar{\pi}\} = \{4^2 31\} \quad \text{from} \quad \{a_0-\bar{\xi}_{\epsilon-0}\}\{1^{r-c_1}\}\{1^{r-c_2}\}\{1^{r-c_3}\} = \{21\}\{1^2\}\{1^3\}\{1^4\}$$

and also how one way of forming $\{\xi\} = \{754^2\}$ from $\{\xi_0\}\{\xi_1\}\{\xi_2\} = \{4^2 32\}\{21\}\{1^4\}$ corresponds to one way of forming $\{2m-\bar{\xi}\} = \{6^2 53\}$ from

$$\{a_0-\bar{\xi}_{\epsilon-0}\}\{a_1-\bar{\xi}_{\epsilon-1}\}\{a_2-\bar{\xi}_{\epsilon-2}\} = \{21\}\{3^2 21\}\{2^4\}:$$



Theorem 33. *$D_{2m-\bar{\lambda}}\{m\} \otimes \{\mu\} = D_{\lambda}\{m\} \otimes \{\mu\}$, where (μ) is any partition of n.*

Proof. It will suffice to show that

$$D_{\lambda}\{m\}^a\,[\{m\}^{(2)}]^b\,[\{m\}^{(3)}]^c \ldots = D_{2m-\bar{\lambda}}\{m\}^a\,[\{m\}^{(2)}]^b\,[\{m\}^{(3)}]^c \ldots.$$

By theorem 31, every decomposition of D_{λ} into $D_{\omega}D_{\nu}D_{\eta}\ldots$, where (ω), (ν), (η), ... are partitions of ma, $2mb$, $3mc$, ... into a, $2b$, $3c$, ... parts respectively, has an exact counterpart in the decomposition of $D_{2m-\bar{\lambda}}$ into $D_{2m-\bar{\omega}}D_{2m-\bar{\nu}}D_{2m-\bar{\eta}}\ldots$. Hence if

$$D_{\lambda}\{m\}^a\,[\{m\}^{(2)}]^b\,[\{m\}^{(3)}]^c \ldots = \sum_{\omega,\nu,\eta,\ldots} g D_{\omega}\{m\}^a\,D_{\nu}[\{m\}^{(2)}]^b\,D_{\eta}[\{m\}^{(3)}]^c \ldots,$$

where g is the coefficient of $\{\lambda\}$ in $\{\omega\}\{\nu\}\{\eta\}\ldots$, then

$$D_{2m-\bar{\lambda}}\{m\}^a\,[\{m\}^{(2)}]^b\,[\{m\}^{(3)}]^c \ldots = \sum_{\omega,\nu,\eta,\ldots} g D_{2m-\bar{\omega}}\{m\}^a\,D_{2m-\bar{\nu}}[\{m\}^{(2)}]^b\,D_{2m-\bar{\eta}}[\{m\}^{(3)}]^c \ldots.$$

But, by theorem 32, $D_{\omega}\{m\}^a = D_{2m-\bar{\omega}}\{m\}^a$, $D_{\nu}[\{m\}^{(2)}]^b = D_{2m-\bar{\nu}}[\{m\}^{(2)}]^b$, and so on. The theorem follows.

Theorem 34. *If $\lambda_2 \leqslant m$ and $m+\lambda_n \geqslant \beta \geqslant \lambda_2$, and $\{\lambda\}'_{\beta}$ denotes*

$$\{2nm-(n-1)\beta-\lambda_1, \beta-\lambda_n, \beta-\lambda_{n-1}, \ldots, \beta-\lambda_2\},$$

then $D_{\lambda}\{\mu\}\{\nu\} = D_{\lambda'_{\beta}}\{\mu\}'_{\beta}\{\nu\}'_{\beta}$, where (λ), (μ), (ν) are partitions of mn, ma, mb into n, a, b parts respectively, and $a+b = n$.

Proof. Consider a rectangle of $n-1$ rows with β nodes in each row. Above this rectangle place a row of λ_1 nodes, starting above the top left-hand node of the rectangle and proceeding from left to right. Below the rectangle place a row of $2nm-(n-1)\beta-\lambda_1$ nodes, starting below the bottom right-hand node of the rectangle and proceeding from right to left. Place the partition graph of (μ) in the top left-hand corner and the inverted graph of $\{\mu\}'_\beta$ in the bottom right-hand corner. Consider in the first place the rectangle only and the construction in it of the diagram associated with $\{\lambda_2, \ldots, \lambda_n\}$ from the product

$$\{\mu_2, \ldots, \mu_a\}\{\nu_1+\mu_1-\lambda_1\}\{\nu_2\}\{\nu_3\}\ldots\{\nu_b\}.$$

To do this, proceed as in theorem 31 by adding $\nu_1+\mu_1-\lambda_1$ symbols α_1 to the diagram of $\{\mu_2, \ldots, \mu_a\}$ so that no two symbols lie in the same column and so that the resulting diagram is, as usual, a Young diagram. Then add ν_2 symbols α_2 in the same way, no two α's with the same suffix lying in the same column, and so on for further sets of $\nu_3, \ldots, \nu_b$ symbols.

In the bottom right-hand corner of the rectangle construct in the same way the diagram of $\{\beta-\lambda_n, \ldots, \beta-\lambda_2\}$ from the product

$$\{\beta-\mu_a, \beta-\mu_{a-1}, \ldots, \beta-\mu_2\}\{\beta-(\nu_1+\mu_1-\lambda_1)\}\{\beta-\nu_2\}\ldots\{\beta-\nu_b\}$$

by applying $\beta-(\nu_1+\mu_1-\lambda_1)$ symbols β_1 to the diagram of $\{\beta-\mu_a, \beta-\mu_{a-1}, \ldots, \beta-\mu_2\}$ so that every column of the rectangle has either an α_1 or a β_1, and then applying $\beta-\nu_2$ symbols β_2 so that every column has either an α_2 or a β_2, and so on. The rectangle will ultimately be filled with the symbols and the diagrams of $\{\mu_2, \ldots, \mu_a\}$ and $\{\beta-\mu_a, \ldots, \beta-\mu_2\}$, thus showing that to each way of building $\{\lambda_2, \ldots, \lambda_n\}$ from $\{\mu_2, \ldots, \mu_a\}\{\nu_1+\mu_1-\lambda_1\}\{\nu_2\}\ldots\{\nu_b\}$ there corresponds one way of building $\{\beta-\lambda_n, \ldots, \beta-\lambda_2\}$ from

$$\{\beta-\mu_a, \ldots, \beta-\mu_2\}\{\beta-(\nu_1+\mu_1-\lambda_1)\}\{\beta-\nu_2\}\ldots\{\beta-\nu_b\}$$

and conversely. The argument of theorem 31 applied to the h-determinantal forms of $\{\nu_2, \ldots, \nu_b\}$ and $\{\beta-\nu_b, \ldots, \beta-\nu_2\}$ then shows that

$$D_{\lambda_2, \ldots, \lambda_n}\{\mu_2, \ldots, \mu_a\}\{\nu_1+\mu_1-\lambda_1\}\{\nu_2, \ldots, \nu_b\}$$
$$= D_{\beta-\lambda_n, \ldots, \beta-\lambda_2}\{\beta-\mu_a, \ldots, \beta-\mu_2\}\{\beta-(\nu_1+\mu_1-\lambda_1)\}\{\beta-\nu_b, \ldots, \beta-\nu_2\}.$$

Since $\lambda_1 \geqslant m$ and $\mu_2 \leqslant m$, then $\mu_2-1-\lambda_1<0$, and so the determinant $D_\lambda\{\mu\}$ whose elements are all D operators will have its first column composed of zeros except for the leading element $D_{\lambda_1-\mu_1}$. Hence it can be written as

$$D_{\lambda_1-\mu_1}D_{\lambda_2, \ldots, \lambda_n}\{\mu_2, \ldots, \mu_a\},$$

and so
$$D_\lambda\{\mu\}\{\nu\} = D_{\lambda_1-\mu_1}D_{\lambda_2, \ldots, \lambda_n}\{\mu_2, \ldots, \mu_a\}\{\nu\}$$
$$= D_{\lambda_2, \ldots, \lambda_n}\{\mu_2, \ldots, \mu_a\}D_{\lambda_1-\mu_1}\{\nu\}.$$

If $\nu_2-1-\lambda_1+\mu_1<0$, then the determinant $D_{\lambda_1-\mu_1}\{\nu\}$, whose elements are S-functions, will have its first column zero but for the leading term $\{\nu_1-\lambda_1+\mu_1\}$, and so

$$D_\lambda\{\mu\}\{\nu\} = D_{\lambda_2, \ldots, \lambda_n}\{\mu_2, \ldots, \mu_a\}\{\nu_1-\lambda_1+\mu_1\}\{\nu_2, \ldots, \nu_b\}.$$

If $\nu_2-1-\lambda_1+\mu_1 \geqslant 0$, then the second element in the first column is non-zero as well as the first, and there will be additional terms in $D_{\lambda_1-\mu_1}\{\nu\}$ due to the product of this second element and its cofactor. It is, however, impossible to construct the diagram $\{\lambda_2, \ldots, \lambda_n\}$ from such terms, since every such term will contain an element $\{\nu_1+r\}$, $r>0$, of the first row of $\{\nu\}$, and $\nu_1+r>\lambda_2$ in virtue of $\lambda_2\leqslant m$ and $\nu_1\geqslant m$. Hence the only effective term in $D_{\lambda_1-\mu_1}\{\nu\}$ is, in all cases, $\{\nu_1-\lambda_1+\mu_1\}\{\nu_2,\ldots,\nu_b\}$ and so $D_\lambda\{\mu\}\{\nu\}=D_{\lambda_2,\ldots,\lambda_n}\{\mu_2,\ldots,\mu_a\}\{\nu_1-\lambda_1+\mu_1\}\{\nu_2,\ldots,\nu_b\}$ in every case.

Similar considerations give

$$D_{\lambda'_\beta}\{\mu\}'_\beta\{\nu\}'_\beta=D_{\beta-\lambda_n,\ldots,\beta-\lambda_2}\{\beta-\mu_a,\ldots,\beta-\mu_2\}\{\beta-(\nu_1-\lambda_1+\mu_1)\}\{\beta-\nu_b,\ldots,\beta-\nu_2\},$$

from which the theorem follows.

As an example, suppose $m=n=7$ and $\{\lambda\}=\{22.6^2543^2\}$, giving $10\geqslant\beta\geqslant6$. Taking $\beta=10$, then $\{\lambda\}'_{10}=\{16.7^2654^2\}$, and if $a=4$, $b=3$, $\{\mu\}=\{18.43^2\}$, $\{\nu\}=\{96^2\}$, then $\{\mu\}'_{10}=\{87^26\}$ and $\{\nu\}'_{10}=\{13.4^2\}$. The diagram is

$$\begin{array}{l}
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \alpha_1\ \alpha_1\ \alpha_1\ \alpha_1\\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \alpha_1\ \alpha_2\ \beta_3\ \beta_3\ \beta_3\ \beta_3\\
\cdot\ \cdot\ \cdot\ \alpha_1\ \alpha_2\ \alpha_3\ \beta_2\ \beta_2\ \beta_2\ \beta_2\\
\cdot\ \cdot\ \cdot\ \alpha_2\ \alpha_3\ \beta_1\ \beta_1\ \beta_1\ \beta_1\ \beta_1\\
\alpha_1\ \alpha_1\ \alpha_1\ \alpha_3\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
\alpha_2\ \alpha_2\ \alpha_2\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
\alpha_3\ \alpha_3\ \alpha_3\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
\beta_1\ \beta_1\ \beta_1\ \beta_1\ \beta_1\ \beta_1\ \beta_1\ \beta_1\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}$$

showing that $\qquad D_{6^25432}\{43^2\}\{5\}\{6\}\{6\}=D_{7^2654^2}\{7^26\}\{5\}\{4\}\{4\}.$

Similarly, $\qquad D_{6^25432}\{43^2\}\{5\}\{7\}\{5\}=D_{7^2654^2}\{7^26\}\{5\}\{5\}\{3\},$

being equal to zero in this example since the diagram cannot be constructed, and so by subtraction
$$D_{6^25432}\{43^2\}\{5\}\{6^2\}=D_{7^2654^2}\{7^26\}\{5\}\{4^2\}.$$

But

$$D_{22.6^25432}\{18.43^2\}\{96^2\}=\begin{vmatrix} D_4 & D_{19} & D_{21} & D_{22} & D_{26} & D_{27} & D_{28}\\ \cdot & D_2 & D_4 & D_5 & D_9 & D_{10} & D_{11}\\ \cdot & D_1 & D_3 & D_4 & D_8 & D_9 & D_{10}\\ \cdot & \cdot & D_1 & D_2 & D_6 & D_7 & D_8\\ \cdot & \cdot & \cdot & 1 & D_4 & D_5 & D_6\\ \cdot & \cdot & \cdot & \cdot & D_2 & D_3 & D_4\\ \cdot & \cdot & \cdot & \cdot & D_1 & D_2 & D_3 \end{vmatrix}\{96^2\}$$

$$=D_4D_{6^25432}\{43^2\}\{96^2\}$$

$$=D_{6^25432}\{43^2\}\begin{vmatrix}\{5\} & \{10\} & \{11\}\\ \{1\} & \{6\} & \{7\}\\ \{0\} & \{5\} & \{6\}\end{vmatrix}=D_{6^25432}\{43^2\}\{5\}\{6^2\},$$

since terms such as $\{1\}\{10\}\{6\}$, $\{1\}\{11\}\{5\}$, $\{0\}\{10\}\{7\}$ and so on, cannot contribute to the building of $\{6^25432\}$. In this example $\{6^2\}$ could be replaced by $\{6\}\{6\}$, since $\{5\}\{7\}\{5\}$ cannot contribute to $\{6^25432\}$, but this need not be so in every case. Similarly,

$$D_{16.7^2654^2}\{87^26\}\{13.4^2\}=D_{7^2654^2}\{7^26\}\{5\}\{4^2\},$$

so that the theorem has been verified.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

THEOREM 35. *If* (μ) *is a partition of* mpr *into* pr *parts and* $m \geqslant \mu_2$, $m + \mu_{pr} \geqslant \beta \geqslant \mu_2$ *and*

$$\{\mu\}'_\beta = \{2mpr - (pr-1)\beta - \mu_1, \beta - \mu_{pr}, \beta - \mu_{pr-1}, \ldots, \beta - \mu_2\},$$

then
$$D_\mu[\{m\}^{(p)}]^r = (-1)^{\beta r(p-1)} D_{\mu'_\beta}[\{m\}^{(p)}]^r.$$

Proof. Following the lines of theorem 32, the sequences $[\alpha_i]$, $[\beta_i]$ for $\{\mu\}$ and $\{\mu\}'_\beta$ respectively are

$$[\alpha_i] = \mu_1 - 1, \mu_2 - 2, \ldots, \mu_{pr} - pr \pmod p$$

and
$$[\beta_i] = \beta - \mu_1 - 1, \beta - \mu_{pr} - 2, \beta - \mu_{pr-1} - 3, \ldots, \beta - \mu_3 + 1, \beta - \mu_2 \pmod p$$
$$= \beta - 2 - (\mu_1 - 1), \beta - 2 - (\mu_{pr} - pr), \beta - 2 - (\mu_{pr-1} - pr + 1), \ldots, \beta - 2 - (\mu_2 - 2).$$

Each of the sequences reduced mod p will thus consist of the residues $0, 1, \ldots, p-1$ with varying multiplicities in general, but if the multiplicities are unequal in $[\alpha_i]$ they will clearly be unequal in $[\beta_i]$ and conversely. It follows that if $\{\mu\} = 0$, then $\{\mu\}'_\beta = 0$ and conversely.

It is now necessary to relate the signs θ and θ' associated with $\{\mu\}$ and $\{\mu\}'_\beta$ in the non-zero case. Let $\alpha_1 = \mu_1 + pr - 1 \equiv y \pmod p$, and $\beta - 2 \equiv \epsilon \pmod p$. Then

$$\mu'_1 = 2mpr - (pr-1)\beta - \mu_1$$
$$\equiv \beta - \mu_1 \pmod p$$
$$\equiv \epsilon - y + 1 \pmod p \quad \text{and so} \quad \beta_1 \equiv \mu'_1 - 1 \equiv \epsilon - y \pmod p.$$

Let $\alpha_i = \mu_i + pr - i \equiv x \pmod p$, for $i > 1$. Then

$$\beta_{pr-i+2} = (\beta - \mu_i) + pr - (pr - i + 2)$$
$$\equiv \beta - \mu_i + i - 2 \pmod p$$
$$\equiv \epsilon - x \pmod p.$$

Now apply the cyclic permutation $(1.2.3.\ldots.pr)$ to the suffixes of $[\beta_i]$. This moves each β_i one place to the left, for $i > 1$, and relegates β_1 to the extreme right-hand place. Then to the first set of p integers from $[\alpha_i]$ which are congruent mod p to $p-1, p-2, \ldots, 1, 0$ in order there now corresponds the last set of p integers from the permuted $[\beta_i]$ which are congruent to $\epsilon - (p-1), \epsilon - (p-2), \ldots, \epsilon - 1, \epsilon$ in order, the suffixes being such that $\alpha_h, \alpha_k, \ldots, \alpha_1, \ldots, \alpha_s, \alpha_t$ correspond to $\beta_{pr+2-h}, \beta_{pr+2-k}, \ldots, \beta_1, \ldots, \beta_{pr+2-t}$.

The second and subsequent sets of p integers from $[\alpha_i]$ which are congruent to $p-1$, $p-2, \ldots, 1, 0$ in order have their counterparts in sets of p integers from $[\beta_i]$ which are congruent to $\epsilon - (p-1)$, $\epsilon - (p-2), \ldots, \epsilon - 1, \epsilon$ in order, the suffixes being such that α_i corresponds to β_{pr+2-i}, the complication of making β_1 correspond to α_1 occurring only in relating the first set from $[\alpha_i]$ with the last from $[\beta_i]$.

It follows that the permutation $(\alpha_1 \alpha_a \alpha_b \ldots)(\alpha_c \alpha_d \ldots) \ldots$ which rearranges $[\alpha_i]$ into r sets, each congruent to $p-1, p-2, \ldots, 1, 0$ in order, corresponds to a permutation of the same order in the cycles which rearranges the permuted $[\beta_i]$ into r sets, each congruent to $\epsilon, \epsilon - 1, \epsilon - 2, \ldots, \epsilon - (p-1)$. This permutation is $(\beta_1 \beta_{pr+2-a} \beta_{pr+2-b} \ldots)(\beta_{pr+2-c} \beta_{pr+2-d} \ldots) \ldots$. The r numbers of $[\beta_i]$ which are congruent to $\epsilon - y$ are not, however, in the required descending order, but they can be put into this order by a cyclic permutation of order r, since the last of them, β_1, has to come into the place of the first of the β_i which is congruent to $\epsilon - y$, and the others have to be moved one place to the right.

All that remains is to convert each of the r sets of integers congruent to $\epsilon, \epsilon-1, \ldots, \epsilon-(p-1)$ into a set congruent to $p-1, p-2, \ldots, 1, 0$. As in theorem 32, this is effected by adding $-(1+\epsilon)$, that is, $-(\beta-1)$, and reducing mod p. The combined effect of the three cyclic permutations on $[\beta_i]$ is incorporated in

$$\theta' = \theta(-1)^{(pr-1)+(r-1)+r(1-\beta)(p-1)} = \theta(-1)^{(p-1)\beta r}.$$

Writing $$\{\mu\} = \theta\{\xi_{01}, \xi_{02}, \ldots, \xi_{0r}\}^{(p)} \ldots \{\xi_{p-1,1}, \ldots, \xi_{p-1,r}\}^{(p)}$$

and $$\{\mu\}'_\beta = \theta'\{\bar{\xi}_{01}, \bar{\xi}_{02}, \ldots, \bar{\xi}_{0r}\}^{(p)} \ldots \{\bar{\xi}_{p-1,1}, \ldots, \bar{\xi}_{p-1,r}\}^{(p)},$$

then $$\alpha_1 = p[\xi_{y1}+r-1]+y,$$

and $$\beta_1 = p[\bar{\xi}_{\epsilon-y,1}+r-1]+(\epsilon-y) \quad \text{if} \quad \epsilon \geqslant y,$$
$$= p[\bar{\xi}_{\epsilon-y,1}+r-1]+p+(\epsilon-y) \quad \text{if} \quad \epsilon < y,$$

where it is assumed that in any suffix, $\epsilon-y$ is replaced by $p+(\epsilon-y)$ whenever $\epsilon<y$.

Let $\beta-2 = \epsilon+pk$; then

$$\alpha_1+\beta_1 = p[\xi_{y1}+\bar{\xi}_{\epsilon-y,1}]+2rp-2p+\epsilon \quad \text{if} \quad \epsilon \geqslant y,$$
$$= p[\xi_{y1}+\bar{\xi}_{\epsilon-y,1}]+2rp-p+\epsilon \quad \text{if} \quad \epsilon < y.$$

But $\alpha_1 = \mu_1+rp-1$ and $\beta_1 = 2mpr-(pr-1)\beta-\mu_1+rp-1$, from which

$$\xi_{y1}+\bar{\xi}_{\epsilon-y,1} = 2mr-r\beta+k+2 \quad \text{if} \quad \epsilon \geqslant y,$$
$$= 2mr-r\beta+k+1 \quad \text{if} \quad \epsilon < y.$$

Next, if $i \neq 1$, then $\alpha_{yi} = p[\xi_{yi}+r-i]+y$ and

$$\beta_{\epsilon-y,r-i+2} = p[\bar{\xi}_{\epsilon-y,r-i+2}+r-(r-i+2)]+\epsilon-y \quad \text{if} \quad \epsilon \geqslant y,$$
$$= p[\bar{\xi}_{\epsilon-y,r-i+2}+r-(r-i+2)]+p+\epsilon-y \quad \text{if} \quad \epsilon < y.$$

But if $\alpha_{yi} = \alpha_j$, then $\beta_{\epsilon-y,r-i+2} = \beta_{pr-j+2}$ and so $\mu_j+pr-j = p[\xi_{yi}+r-i]+y$ and

$$\beta-\mu_j+j-2 = p[\bar{\xi}_{\epsilon-y,r-i+2}+i-2]+\epsilon-y \quad \text{if} \quad \epsilon \geqslant y,$$
$$= p[\bar{\xi}_{\epsilon-y,r-i+2}+i-2]+p+\epsilon-y \quad \text{if} \quad \epsilon < y,$$

from which, when $i \neq 1$, $$\xi_{yi}+\bar{\xi}_{\epsilon-y,r-i+2} = k+2 \quad \text{if} \quad \epsilon \geqslant y,$$
$$= k+1 \quad \text{if} \quad \epsilon < y.$$

In the same way it can be shown that when $x \neq y$,

$$\xi_{xi}+\bar{\xi}_{\epsilon-x,r-i+1} = k+1 \quad \text{if} \quad \epsilon \geqslant x,$$
$$= k \quad \text{if} \quad \epsilon < x,$$

for $i = 1, 2, \ldots, r$.

If $a_x = k+1$ when $\epsilon \geqslant x$, and $a_x = k$ when $\epsilon < x$, and $a_y = k+2$ when $\epsilon \geqslant y$, and $a_y = k+1$ when $\epsilon < y$, then

$$a_0+a_1+a_2+\ldots+a_{p-1} = (\epsilon+1)(k+1)+(p-1-\epsilon)k+1 = \beta.$$

It has thus been shown that if

$$\{\mu\} = \theta\{\xi_{01}, \xi_{02}, \ldots, \xi_{0r}\}^{(p)} \ldots \{\xi_{y1}, \xi_{y2}, \ldots, \xi_{yr}\}^{(p)} \ldots \{\xi_{p-1,1}, \ldots, \xi_{p-1,r}\}^{(p)},$$

then
$$\{\mu\}'_\beta = (-1)^{\beta r(p-1)}\theta\{a_0-\xi_{0r}, a_0-\xi_{0,r-1}, \ldots, a_0-\xi_{01}\}^{(p)} \ldots$$
$$\ldots\{2mr-r\beta+a_y-\xi_{y1}, a_y-\xi_{yr}, a_y-\xi_{y,r-1}, \ldots, a_y-\xi_{y2}\}^{(p)} \ldots$$
$$\ldots\{a_{p-1}-\xi_{p-1,r}, a_{p-1}-\xi_{p-1,r-1}, \ldots, a_{p-1}-\xi_{p-1,1}\}^{(p)},$$
where $a_0+a_1+\ldots+a_{p-1}=\beta$.

It remains to show that if $(\xi)=(\xi_1,\xi_2,\ldots,\xi_r)$ is a partition of mr, then the coefficient of $\{\xi\}$ in
$$\{\xi_{01},\xi_{02},\ldots,\xi_{0r}\}\ldots\{\xi_{y1},\ldots,\xi_{yr}\}\ldots\{\xi_{p-1,1},\ldots,\xi_{p-1,r}\}$$
is equal to the coefficient of
$$\{\xi\}'_\beta = \{2mr-(r-1)\beta-\xi_1, \beta-\xi_r, \beta-\xi_{r-1}, \ldots, \beta-\xi_2\}$$
in the product of the corresponding S-functions associated with $\{\mu\}'_\beta$.

The argument of the last part of theorem 32 will apply here with only a slight modification, for when $x \neq y$, the diagrams of
$$\{\xi_{x1},\xi_{x2},\ldots,\xi_{xr}\} \quad\text{and}\quad \{a_x-\xi_{xr}, a_x-\xi_{x,r-1}, \ldots, a_x-\xi_{x1}\}$$
can be combined into an $a_x\times r$ rectangle, whilst
$$\{\xi_{y1},\xi_{y2},\ldots,\xi_{yr}\} \quad\text{and}\quad \{2m-r\beta+a_y-\xi_{y1}, a_y-\xi_{yr}, \ldots, a_y-\xi_{y2}\}$$
give an $a_y\times r$ rectangle with the first row continued $\xi_{y1}-a_y$ places to the right, and the last row continued $2mr-r\beta-\xi_{y1}$ places to the left.

It is convenient then to proceed as in theorem 32 until the $\beta\times r$ rectangle is filled up, taking $\{\xi_{y1},\xi_{y2},\ldots,\xi_{yr}\}$ and its counterpart as the final members of the products giving $\{\xi\}$ and $\{\xi\}'_\beta$ respectively. Those nodes falling outside the $a_y\times r$ rectangle will then necessarily fall outside the $\beta\times r$ rectangle, giving the first rows of $\{\xi\}$ and $\{\xi\}'_\beta$ respectively. Thus to each way of constructing $\{\xi\}$ from the first product there is a unique way of constructing $\{\xi\}'_\beta$ from the second product and conversely.

It has thus been shown that if
$$D_\mu = \theta[\sum_\xi g_{\mu\xi}D^{(p)}_\xi], \quad\text{then}\quad D_{\mu'_\beta} = (-1)^{\beta r(p-1)}\theta[\sum_\xi g_{\mu\xi}D^{(p)}_{\xi'_\beta}].$$
But from theorem 2, corollary 2,
$$D^{(p)}[\{m\}^{(p)}]^r = D^{(p)}_{\xi'_\beta}[\{m\}^{(p)}]^r,$$
since the presence of the superscripts is immaterial, and so the proof is complete.

THEOREM 36. *If $\lambda_2\leqslant m$, and $m+\lambda_n\geqslant\beta\geqslant\lambda_2$, then*
$$D_{\lambda'_\beta}\{m\}\otimes\{\mu\} = D_\lambda\{m\}\otimes\{\mu\} \quad \textit{for even } \beta,$$
$$= D_\lambda\{m\}\otimes\{\tilde{\mu}\} \quad \textit{for odd } \beta,$$
where (μ) is any partition of n.

Proof. From theorem 34, if
$$D_\lambda\{m\}^a[\{m\}^{(2)}]^b[\{m\}^{(3)}]^c\ldots = \sum_{\omega,\nu,\eta,\ldots} gD_\omega\{m\}^a D_\nu[\{m\}^{(2)}]^b D_\eta[\{m\}^{(3)}]^c\ldots,$$
where (ω), (ν), (η), ... are respectively partitions of ma, $2mb$, $3mc$, ... into a, $2b$, $3c$, ... parts, and g is the coefficient of $\{\lambda\}$ in $\{\omega\}\{\nu\}\{\eta\}\ldots$, then
$$D_{\lambda'_\beta}\{m\}^a[\{m\}^{(2)}]^b[\{m\}^{(3)}]^c\ldots = \sum_{\omega,\nu,\eta,\ldots} gD_{\omega'_\beta}\{m\}^a D_{\nu'_\beta}[\{m\}^{(2)}]^b D_{\eta'_\beta}[\{m\}^{(3)}]^c\ldots.$$

But by theorem 35, $D_\omega\{m\}^a = D_{\omega'_\beta}\{m\}^a$,

$$D_\nu[\{m\}^{(2)}]^b = (-1)^{\beta b} D_{\nu'_\beta}[\{m\}^{(2)}]^b, \quad D_\eta[\{m\}^{(3)}]^c = D_{\eta'_\beta}[\{m\}^{(3)}]^c,$$

and so on. Hence if β is even, $D_\lambda\{m\}^a[\{m\}^{(2)}]^b \ldots = D_{\lambda'_\beta}\{m\}^a[\{m\}^{(2)}]^b \ldots$, whereas if β is odd the sign is reversed whenever the class $1^a2^b3^c \ldots$ is an odd class. The theorem follows.

Theorems 28, 29, 30, 33 and 36 are of considerable use in the computation of $\{m\}\otimes\{\mu\}$. They have been used in conjunction with one of Littlewood's (1944) methods to compute the full expansion when $m = 5$ and (μ) is any partition of 5.

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